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THE

TEACHING OF MATHEMATICS IN THE ELEMENTARY AND THE SECONDARY SCHOOL

BY

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ASSOCIATE PROFESSOR OF THE PEDAGOGY OF MATHEMATICS
IN THE UNIVERSITY OF CHICAGO

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Preface

I SHOULD have been untrue to my profound conviction of the need of pedagogic preparation antecedent to entrance upon the actual work of teaching mathematics if I had not kept *prospective* teachers of mathematics in mind throughout the preparation of this book; I should have ignored the patent fact that it is experienced teachers who feel most keenly the problems of teaching and the need for their study if I had addressed myself to the inexperienced exclusively. As a consequence, unity of treatment and homogeneity of style have been in a measure sacrificed in what follows; the experienced teacher will, according to his experience, find this or that trite and superfluous; the inexperienced aspirant will occasionally feel the lack of the perspective of experience.

I appreciate deeply the importance of the historical aspect of the teaching of mathematics, but in view of its genial treatment in the excellent and well-known work of Smith, I have felt at liberty not to touch upon it. Some writers on methods in mathematics discuss more or less extensive topics of subject matter, in the form either of presentations to pupils or of a more intensive study from the teacher's viewpoint. Such work is important — an essential element of progress, indeed; but I have found it advisable here strictly to confine myself to illustrative use only of subject matter. Within the bounds of the field thus limited I have further restricted myself, in the main, to conditions as they exist in the United States.

I have naturally expressed my own opinions freely, but I have endeavored also to give place to important different points of view, either voicing them by direct quotations, indicating them by references, or mentioning works containing them in the bibliography. I need hardly say that while in the attempt to speak clearly and to the point I have stated my views without circumlocution, they are expressed without the slightest spirit of dogmatism. Most of the material has been used repeatedly in classes, and this may in some measure account for a too didactic style.

The bibliographies had their origin in reference lists of books and papers accessible to members of my classes, and I have enlarged the lists from the memoranda of my own reading. With the exception of a few papers which I knew to be in the printer's hands, no references are given to publications later than February, 1906. That lists so made will be guilty of many sins of omission and of commission is axiomatic, but I trust that nevertheless the bibliographies and other references may serve to turn in the right direction the inquirer who is seeking the best of the current literature on the various subjects. In view of the fact that one of the chief functions of a book like the present is that of a work of reference, I have not hesitated at frequent repetitions, both in the bibliographies and also in the text itself.

J. W. A. YOUNG

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The Teaching of Mathematics

CHAPTER I

THE STUDY OF THE PEDAGOGY OF MATHEMATICS

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Burkhardt, Encyklopädie der Mathematik, Leipzig, now appearing, is also to have a division on the pedagogy of mathematics.

Articles on mathematical topics are found in various cyclopedias of education. The most important of these is:

Rein. Handbuch der Pädagogik. 16 vols. Langensalza. Second edition now appearing; first edition exhausted.

Mention may also be made of:

Buisson, F. Dictionnaire de Pédagogie et d'Instruction primaire. 4 vols. Paris, 1887.

Kiddle and Schem. Cyclopedia of Education. Third Edition. New York, 1876.

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Schmid, K. A. Encyklopädie des gesammten Erziehungs- und Unterrichtswesens. Second Edition, 10 vols. Leipzig, 1887.

Rein, Pickel, und Schiller. Theorie und Prakis des Volksschul-Unterrichts nach Herbartschen Grundsätzen. New Edition, 8 vols. Dresden, 1888.

Some of the general cyclopedias, in particular the Encyclopedia Britannica (tenth edition), contain good articles on mathematical topics.

The Penny Cyclopedia has valuable articles by De Morgan.

The Study needed.

THAT the prospective teacher of mathematics should prepare himself for his work by some theoretic and practical study, under guidance, of the methods of the art which he expects to exercise, would seem to be an entirely superfluous statement, were it not the fact that only a small minority of those now teaching mathematics in this country have had any such preparation. No physician would be considered sufficiently prepared to exercise his art by the mere study of theoretic books, and the recollection of how he was treated by his family physician in his own illness; no one would think of entrusting his watch for repairs to a man who had never been shown the construction of a watch, and had never actually taken apart and put together watches; even the artisan must serve his apprenticeship. Yet it has not seemed astounding that young men and women equipped only with a more or less adequate knowledge of the subject matter, and some general recollections of how they themselves were taught, should be given precious minds to make or to mar.

The Growth of the Interest in General Pedagogy.

The past few decades have witnessed a remarkable growth in the attention paid to general pedagogy, especially in America. Few of the larger institutions are now without a department of pedagogy, and the progress in the study of general aspects of pedagogy is leading naturally to the study of its phases as modified by the needs of particular subjects.

The more thorough study of pedagogic questions has borne tangible fruit in numerous proposals for improvement. Changes, reforms are in the air. Where central authorities promulgate curricula for an entire country, as in France and Germany, several markedly different programmes have been established during the last quarter-century, the most recent in each of the two countries named bearing a twentieth-century date. While the most conspicuous changes usually relate to languages, ancient and modern, the mathematical programmes have also undergone noteworthy changes.

The Pedagogic Unrest.

In mathematics, perhaps more than in any other subject, the opinion has been current that a knowledge of the subject matter is a sufficient preparation for the teacher. This pedagogic unpreparedness, often combined with a very mediocre grasp of the subject matter, has led to not a little teaching by rote, and is responsible for much of the disproportion between time spent and results achieved in our American mathematical instruction.[1] It is even not unknown that classes in mathematics have been confided to teachers of other subjects, having neither special preparation for teaching mathematics nor experience in it, for no other reason than that they had a vacant period. Happily, however, there is a growing appreciation of the fact that mathematics is probably the most difficult of all subjects to teach, and that its teaching should not be undertaken without thorough preparation not only in the subject matter but also in the art of presenting it.

The Pedagogy of Mathematics.

The questions relative to the teaching of elementary mathematics are receiving world-wide attention in our day, and nowhere has so high a standard for the preparation of teachers been put into practical effect as in Germany.

German Tendencies.

Germany is the home of pedagogy. Here, for perhaps the longest time and with the most thoroughness, have the problems of pedagogy been studied. This study has long since led the Germans to see the importance of careful pedagogic training following a thorough grounding in the subject matter of mathematics itself, and they have

[1] For comparison between American and Prussian results see Young, *The Teaching of Mathematics in Prussia*, New York, 1900, pp. 106–111.

acted upon their convictions by requiring of all aspirants to the ranks of secondary teachers at least three years of university study followed by a year of pedagogic training. And recent decades and recent years have seen a marked tendency in Germany to lay even more stress on pedagogic preparation.[2] This may be seen, for example, in the lengthening of the term of pedagogic training from one year to two, in the organization of new seminaries for the more effective conduct of this training, and in a tendency towards a change of attitude on the part of the mathematical departments of the universities, which, notwithstanding the fact that a large part of the students of mathematics are preparing to teach the subject, have in general paid no attention to its pedagogic aspects. But a distinct tendency is now noticeable to take some account of the prospective profession of the students, and in this and other ways to strengthen still more the pedagogic preparation of the teacher. In this movement Klein, who has no doubt exercised a stronger personal influence on the development of American mathematics and mathematicians of the present day than any other living European, is taking the lead.[3] His attitude may be seen in a remark [4] about Schellbach, the greatest worker in the pedagogy of mathematics Germany has yet seen. "The secret of the influence of Schellbach lay not only in the breadth of his scientific view, but essentially in his eminent pedagogic gifts, and one must concede that successful teaching of mathematics in the school

[2] We are speaking here simply of the pedagogic preparation of teachers. The German movement has a much wider scope and is imbued with the same spirit which will be discussed more fully in Chapter VI.

[3] See Rethwisch, *Jahresbericht ü höh. Schulen*, 1895. Math. 2. Klein, *Ueber eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höherens Schulen*, Leipzig, 1904, pp. 82. In this work references are given to Klein's publications during the last decade bearing on the teaching of elementary mathematics, and several are reprinted.

[4] In Lexis: *Reform der höh. Schulen in Preussen*, Halle, 1902, p. 258.

is perhaps more a matter of art than of science. This art is in recent years being itself made the subject of scientific study."

Progress in England and America.

In England, also, the study of the teaching of secondary mathematics has taken on new life in recent years. The British Association for the Advancement of Science, the leading scientific society of the country, has held important and fruitful discussions on the teaching of mathematics, influential committees of university and school men have drawn up valuable and practical reports, the examining bodies have made the desired modifications in their standards, and as the result of all this there has taken place what it is not an exaggeration to call a revolution in the teaching of geometry in England.[5]

America is likewise bestirring herself. Beginning with the Central Association of Science and Mathematics Teachers, formed at Chicago in 1902, and stimulated by the timely address of the President of the American Mathematical Society at the close of that year,[6] urging that the body of experts in mathematics, not only university and college men as represented in the membership of the society, but secondary teachers as well, should unite in organized and continuous attention to the questions of improvement of education in mathematics, enthusiastic and promising societies for the study of these questions have been springing up in rapid succession in nearly all parts of the United States,[7] culminating in the organization in 1905, by a body of delegates from the existent societies, of a national society for the study of questions relative to the teaching of mathematics and the physical sciences.

In the organization of these societies, in the courses on the pedagogy of mathematics offered in increasing numbers by our

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[5] References to some of the publications of this movement will be given in the bibliography of Chapter VI.

[6] Moore, Presidential address on "The Foundations of Mathematics." *Bull. Am. Math. Soc.* 1903, p. 422. (Also published elsewhere; for references see Chapter VI.)

[7] A list of these societies with their officers is published monthly in *School Science and Mathematics*, Chicago.

leading universities and colleges, and in the recent founding of institutions for the training of secondary teachers, the growth of the conviction that better pedagogic preparation of secondary teachers is needed, may be clearly read.

The Outcome.

It is at present too early to predict what will be the outcome of the international movement for improvement in the teaching of mathematics in the midst of which we now stand. It may at least be regarded as a great step forward that the feeling of unrest exists; that the need for improvement is widely recognized; that the calm feeling that mathematics is a finished subject, that its teaching is subject to no further improvement, has been effectually dissipated. The pendulum is swinging away from the abstract, the formalistic, the self-satisfied, — possibly even to the other extreme, — at any rate it is swinging, and the day of comparative readjustment is not yet.

The Published Results.

The published results of the work done in the pedagogy of mathematics during the last few decades are scattered through many journals and text-books in several languages, and little has been done as yet in the way of collecting these results and presenting them in connected form. To attain a moderately satisfactory survey of them is a laborious, time-consuming and difficult task. The principal general works on the subject are mentioned in the bibliography above, but in only one of them [8] have references to the literature been systematically given.

Relation of the Pedagogy of Mathematics to other Subjects.

The pedagogy of mathematics stands in close relation to general pedagogy, to psychology, to philosophy, to logic. It might seem that these other subjects should be taken up first, since they find their application in the pedagogy of mathematics. But in accordance with the sound principle, *from the less to the more general*, there can be no objection to beginning with the study of the pedagogy of mathematics and following its lead to the other fields. This may be the case at least with those whose

[8] Smith, *Teaching of Elementary Mathematics.*

dominant interest is in the teaching of mathematics. If the subject of direct interest is taken up first, interest in the other subjects will be aroused as soon as they are found to be essentially involved in a thorough understanding of the pedagogy of mathematics. No antecedent work in these other fields has, therefore, been presupposed in what follows; but likewise no hesitancy has been felt in entering paths which lead away from mathematics proper, and in following them far enough to give a start to those who may wish to penetrate further. Those who seek only consideration of questions of mathematics and its pedagogy in the most restricted sense have but to resist the temptations to wander away.

Dangers of the Study of Pedagogy.

Mention should also be made of the objections which have been urged against the formal study of pedagogy. For example:

"In the first place, I have my full share of the prejudices created against 'methods' by the superficial, ill-balanced work of the early normal schools. In the second place, I hold that the student who has been well taught has necessarily had, along with his conscious instruction in the science of physics, a good deal of possibly unconscious instruction in the art of teaching physics. In the third place, I have some apprehension lest the conscious study of this art will be accompanied by an overconscious attention to the philosophy and psychology of the art, with the possible result of setting up a more ponderous system of mental machinery than can be used to advantage in the very practical, common-sense business of teaching young people."[9]

To this one might reply:

1. Wherever good work is done to-day, it should be judged on its own merits, without prejudice arising from bad work of yesterday.

2. Even if all that is urged above be granted, the systematic study of the science and art of teaching is still none the less desirable. The point of view of one avowedly preparing to teach is vastly different from that of the pupil,

[9] Hall, *Teaching of Physics*, New York, 1902, p. 244.

engrossed by the subject matter, and properly unconscious of himself as a pedagogic problem. What he may unconsciously and accidentally absorb is vastly different from what he could learn in somewhat maturer years under the well-planned guidance of an experienced teacher.

3. The objection urging the danger of "philosophizing" and of resultant machinery is possibly the strongest, but it may be questioned whether it is knowledge or *little* knowledge that is the dangerous thing, and whether the remedy does not rather lie in a more thorough and scientific study of the problems of teaching than in ignoring them altogether.

Learn to teach by teaching.

Another objection to the theoretic study of pedagogy is based on the assertion that teaching is an art which can be learned *only* in its actual exercise, and there is some plausibility in the assertion. Actual teaching is beyond question the most effective of all ways of learning the art, but it does not follow that it should not be preceded by special preparatory instructions as to the work to be done, or that the first teaching should not be attempted under the sympathetic guidance of teachers of experience, present and assisting at the time and suggesting improvements afterwards. This variety of learning by practice is far removed from turning the novice loose upon helpless children with nothing but his inexperience to guide him.

How the teaching of Mathematics may be studied.

There are various ways in which teaching mathematics may be studied with profit:

1. By reading the published results of the experience of others.
2. By personal consultation with experienced teachers.
3. By observation of teachers at work.
4. By actual teaching.

The best-arranged schemes of training in the art of teaching include considerable work under each of the four heads, which are arranged in order of increasing importance. Quite a little work should be done under the first three heads before the fourth and chief is taken up.

CHAPTER II

THE PURPOSE AND VALUE OF THE STUDY OF MATHEMATICS IN PRIMARY AND SECONDARY SCHOOLS

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What ought Study of Mathematics to contribute to the Education of High-school Pupil? SCHOOL REVIEW, p. 105. 1898.

The Question

The Need of considering the Question.

THOSE who consider the teaching of any subject are confronted at the very outset by the question, "What is the real purpose and value of the teaching of this subject?" The teacher of mathematics may not feel absolved from giving this question thoroughgoing consideration, by the fact that his subject has long held an honored place in the curriculum and that its value is generally recognized. To make clear to others the value of one's subject may be an occasional need, but it is a permanent and fundamental need that every teacher make the function of his subject in the curriculum clear to himself, and that he keep it constantly before his mind as the determining motive of all his work.

The interests of good teaching demand that the teacher know not merely *what* to teach, and *how* to teach, but *why* he teaches.[1] In fact he must first of all know why he teaches before he can determine intelligently either what or how to teach.

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[1] It is true that what is to be taught is determined in the first instance, and in a general way, by the makers of curricula and those who select the text-books. Still the teacher has more or less leeway as to what he teaches and he determines the mode of teaching almost entirely.

The best answer to the question can be found only by approaching it in a broad spirit; the subject must be regarded not as isolated but as part of a general scheme of education. It is not a sufficient aim to seek to "cover" a specific list of facts, or as many facts as possible in a given time, or even to develop as far as may be the power of independent thought along the lines of the subject in hand. In other words, unless specialists are being trained, no subject may be regarded as entirely self-centered. No subject, unless it be the mother tongue as the vehicle of all knowledge and the medium of all intercourse, can justly be made an essential element of every primary and secondary school course, an inevitable requirement for every pupil, solely on account of its own content. Each subject has a far broader function, and if the teacher has before his mind no other ideal than to make his pupils masters certain subject matter, of a some of the best results which might be achieved will fail of realisation.

The Attitude of the Pupil.

Not only is it the duty of the teacher to know why his subject is a part of the curriculum, but it is the privilege of the pupil to consider the same question.

Some pupils no doubt regard the whole process of education (or any particular subject) as a set of tasks intended in some undefined way for the gratification of others, and consider that their own best interests lie in evading as far as possible the execution of these tasks. Others realize, more or less clearly and consciously, that their own good is the chief end in view, and in customary submission to the authority of parents and teachers unquestioningly traverse the path marked out for them. Still others may go a step further and ask how education in general, or some branch of study in particular, will prove of value to them.

Such an inquiry, if brought to parents or teachers, deserves a cordial reception. Every indication that the mind is being roused from a state of passive receptivity to active inquiry is a mark of progress, and all such tendencies should be fostered, no stone being left unturned to secure the intelligent co-operation of the pupil, rather than his unwilling or even willing acquiescence.

The pupil's experience has, of course, been too limited to enable him to grasp the *full* force of all the reasons which appeal to the teacher, but with this fact duly emphasized the inquiring pupils may be freely told all the considerations which lead the teacher to believe in the value of his subject, and which influence his teaching. The pupil will usually appreciate enough of what is laid before him to recognize the importance and value of the subject under discussion. If, in rare instances, a sincere inquirer is still in doubt he will at least feel that his teacher has been candid with him, and is not treating him like an unreasoning animal that is arbitrarily put through a certain course of training. The pupil is the one whose welfare is at stake in the process of education, and if he so desires he is entitled to be told as much as he can comprehend of its uses and purposes. The parents of the pupils and the friends of education, if interested, are also entitled to know the purpose and value of the teaching. These general remarks apply with especial force to mathematics whose educational usefulness may be called into question on account of its abstract character and technical form. The teacher of mathematics must therefore be ready to present on occasion a cogent statement, in suitable form, of the purpose and value of the study of mathematics in primary and secondary schools.

Such a discussion can be definite only in so far as it is based on a clear idea of the aim and character of the entire process called *education*. This process has been described in many ways.[2]

What is Education?

These descriptions are usually based upon the thought that education aims primarily to develop the powers of the pupil.

[2] For example:

"*Evidences of Education:*

"1. Correctness and precision in the use of the mother tongue.

"2. Those refined and gentle manners which are the expression of fixed habits of thought and action.

"3. The power and habit of reflection by which the mind is

Power to think and to do is the ultimate end; the acquisition of facts is rather one of the instrumentalities used in the attempt to develop this power than itself the end of education.

In what way is mathematics fitted to bear part in the process? What can mathematics contribute to the result? What can it *alone* contribute? What can be accomplished better by means of mathematics than otherwise? These are the questions to be considered.

The Practical Value of Mathematics

The Facts of Mathematics.

Before taking up these questions, brief mention may be made of the importance of the facts of mathematics, especially since many measure the importance of the study of any subject largely by the importance of its facts. There is no subject, except the use of the mother tongue, which is so intimately connected with everyday life, and so necessary to the successful conduct of affairs. Wherever we turn in these days of iron, steam and electricity, we find that mathematics has been the pioneer and guarantees the results. Were its backbone of mathematics removed, our material civilization would inevitably collapse.

But widespread as are the applications of mathematics and enormous as is its practical value, it may be justly urged that to the large majority of people its importance, though great, is *indirect*, and that the average citizen has but little need of mathematical facts, or even opportunity to use them beyond the merest elements of arithmetic. This is undoubtedly true, though the remark would apply with equal force to every other subject of study. The elements of English (but only the elements) are in constant use; and the elements of mathematics are in occasional use; of all the other subjects taught in school, it is difficult to find any of which use is made in the

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taught to answer questions, as How and Why, through science and philosophy.

"4. The power of growth.

"5. The power to do." — N. M. Butler, *Educational Review*. Nov. 1901.

occupations of after life by substantially all the pupils. While there are few occasions on which the ability to solve a quadratic equation is directly useful in business, it would be hard to imagine any where the ability to scan hexameters, to name the parts of a flower, or to give the date of the battle of Waterloo would be of practical benefit.

The Contingent Practical Value.

A subject is also valuable as preparation for the contingency that the child in the future may take up an occupation requiring knowledge of the subject in question. For mathematics this value is marked, because there is a large and growing number of occupations which require knowledge of mathematical results.

The Informational Value.

Besides the practical values, both certain and contingent, subjects have value on account of the information they impart. This is a very important reason for the study of geography, history and literature. It is an equally strong reason for the study of mathematics. Mathematics is a type of thought which seems ingrained in the human mind, which manifests itself to some extent with even the primitive races, and which is developed to a high degree with the growth of civilization. And in whatever civilization it may be found, the mathematics is essentially the same. It may be of different scope, but is always of the same character. So far as the same ground has been covered the same result has been reached. One nation has not found that the $6 \times 7 = 42$, while another has found that $6 \times 7 = 43$; one age has not found that the square on the hypothenuse of a right triangle equals the sum of the squares on the other two sides, while another found it to equal twice the sum. The Hindus formulated and solved more than one mathematical problem which the Europeans took up and solved independently centuries later, only to learn centuries later still that an earlier civilization had solved them long before.

A type of thought, a body of results, so essentially characteristic of the human mind, so little influenced by environment, so uniformly present in every civilization, is one of which no well-informed mind to-day can be ignorant.

Mathematics is equally ingrained in nature, at least in nature as seen and interpreted by the human mind. The study of nature leads to weighing and measuring, and the establishing of relations which can be expressed in mathematical form and hence studied by mathematical methods. In the large majority of instances these measurements are *indirect*, that is, some related quantity is measured directly and the value desired is computed from it. This at once brings into play a considerable amount of geometric and algebraic theory. The most salient feature of natural phenomena is change, variation; the most important single branch of mathematics — the Calculus — is a study of variation, and may be called, in an important sense, the mathematics of nature. Geometry is an outgrowth of field measurements, as its name implies, and in fact there is little of secondary mathematics, at least, that might not have come into existence as direct or indirect consequence of mathematical formulation of the quantitative relations which exist in nature.[8]

Mathematics in Nature.

So completely is nature mathematical that some of the more exact natural sciences, in particular astronomy and physics, are in their theoretic phases largely mathematical in character, while other sciences which have hitherto been compelled by the complexity of their phenomena and the inexactitude of their data to remain descriptive and empirical, are developing towards the mathematical ideal, proceeding upon the fundamental assumption that mathematical relations *exist* between the forces and the phenomena, and that nothing short of the discovery and formulations of these relations would constitute definitive knowledge of the subject. Progress is measured by the closeness of the approximation to this ideal formulation.

[8] "The theory most prevalent among teachers is that mathematics affords the best training for the reasoning powers; and this in its traditional form. The modern, and to my mind the true, theory is that mathematics is the abstract form of the natural sciences; and that it is valuable as a training of the reasoning powers, not because it is abstract, but because it is a representation of actual things." — Safford, *Mathematical Teaching*, 1886, p. 9.

Little can be understood of even the simplest phenomena of nature without some knowledge of mathematics, and the attempt to penetrate deeper into the mysteries of nature compels simultaneous development of the mathematical processes.[4] Many of the topics of mathematics, from the simplest to the most abstract, have been formulated and treated in direct consequence of the exigencies of the study of nature.[5]

It is evident that the pupil in the secondary school cannot thoroughly appreciate the mathematical constitution of nature, but he can at least catch some glimpses of it, and in portions of physics he may get a clear view of the essentially mathematical character of the phenomena studied.[6]

Pedagogic Bearing.

In consequence of the practical importance of the facts of mathematics, second in actual experience to the mother tongue alone, potent means of arousing and holding the pupil's interest may be found in the practical side of mathematics; in its constant presence in the phenomena of nature,[7] in the many occupations and practical conditions

[4] "Beyond the microtome, the microscope, the statistics of observation, of experiment, of what instrument of world-conquest must the new science avail herself? The answer is patent: of mathematics, that giant pincers of scientific logic which showed Newton the moon simply as a bigger apple trying to fall down on his head, flashed out in the mind of Adams the unseen planet Neptune, told Rayleigh that the chemists had always been breathing vast quantities of argon without knowing it, pointed to Mendelieev the places of unknown chemical elements. And through Helmholtz and his pupil Hertz it has given us the Lenard rays, the Roentgen rays, radium itself, and wireless telegraphy based on Hertzian waves." — Halsted, Biology and Mathematics, *Science*, 1905, p. 161.

[5] A very important end of the study of mathematics is "to foster the conviction that correct thought on the basis of correct premises gives mastery of the external world." — Klein.

[6] "Mathematics stands forth as that which unites, mediates between Man and Nature, inner and outer world, thought and perception, as no other subject does." — Froebel. Trans. Herford. London, 1893, Vol. I., p. 84.

[7] "Mathematics in its pure form, as arithmetic, algebra, geometry, and the application of the analytical method, as well as mathe-

requiring knowledge of mathematics, in the possibility of its direct outgrowth from the pupil's own motor activity.[8]

Mathematics as a Mode of Thought

The Chief Value of the Study of Mathematics. But the facts of mathematics, important and valuable as they are, are not the strongest justification for the study of the subject by all pupils. Still more important than the subject matter of mathematics is the fact that it exemplifies most typically, clearly and simply certain modes of thought which are of the utmost importance to every one.

Grasping a Situation. One of these modes of thought is the ability to grasp a situation, to seize the facts, and to perceive correctly the state of affairs. This is prerequisite to success in any occupation, but it is a hard thing to do in actual life. The real facts, must often be sifted out with great care. A physician must contend with the inability of the patient to understand or to describe his affliction; the crucial fact may often be buried under a mass of trivial or irrelevant details. The business man must constantly reckon with a positive and active attempt to mislead him. He must learn the facts, if he can, not only without assistance, but in spite of active opposition. The web of facts and relationships in which every one moves is moreover extremely tangled. Much practice is requisite to even fair success in grasping situations, and we look to the school to furnish and direct such practice. Mathematics is specially adapted to the *beginning* of the practice, because its facts are few and uncomplicated. The situations which it presents can be made very simple at first; it is possible for both pupil and teacher to *know* whether or not the

matics applied to matter and force, or statics and dynamics, furnishes the peculiar study that gives to us, whether as children or as men, the command of nature in this its quantitative aspect; mathematics furnishes the instrument, the tool of thought which we wield in this realm." — Harris, W. T., *Psychologic Foundations of Education*, New York, 1898, p. 325.

[8] See chapters on Laboratory Method and on Arithmetic.

pupil has really grasped them, and they can be made more complicated as the pupil's ability increases.

It is a long way from the simple system of elementary truths of mathematics, whose present form is the result of centuries of polishing and remodelling, to the complex of facts of the social organism. Mathematics can consequently only make the beginning in training for the inevitable grapple with the facts of the world: other subjects growing gradually more complicated must take it up, indeed some others, for example the simplest phases of nature study and of elementary physical science, may be equally definite and clear cut, and may well go hand in hand with mathematics, but it is only in their simplest stages that these subjects retain the precision which on the other hand characterizes mathematics throughout.

Pedagogic Bearing.

What has preceded evidently demands that the entire course of instruction tend to make the pupil appreciate clearly and quickly what is given and what is to be found and done. This feature can hardly be made too prominent. Much of the lack of success which some pupils experience is due to weakness at this point. The pupil plunges in headlong to do *something* at all hazards, instead of holding himself in hand and first deliberating on what he is to do and what materials and tools he has with which to do it. The explicit statement of what is granted (the hypothesis) and what is to be done (the conclusion) may on suitable occasion be made matter of special drill. The Q. E. D. of mathematics is also of great importance. At the beginning, the explicit formulation of what is known and what is required; at the end, the explicit pointing out that the requirements have been met.

To get the pupil to do this habitually and well is not an easy task. Much drill is needed, and the teacher may count it a great step forward when he has brought his pupils to this point. Frequent rapid drills on review matter will be useful where accuracy and speed are the sole ends in view. Drills for understanding what is said are also useful. Short, timed, written exercises may often be used for the same purpose

One of the good features of the English examination system is that it necessitates quick laying and carrying out of plans.

Perhaps through mistaken kindliness teachers sometimes fail to give the pupil the best training in this respect. The beginner, especially if timid, needs encouragement to try, even despite mistakes, but blind guessing and haphazard manipulation should never be encouraged, and when recognized beyond question are not to be tolerated for a moment. It is also mistaken kindliness to repeat two or three times what with wide-awake attention should be grasped the first time.

When the situation is grasped, when the facts are as well in hand as possible, inferences must be made from them, conclusions must be drawn.

Drawing Conclusions.

All thinking and all actions are influenced by conclusions that have been consciously or unconsciously drawn. This act is fundamental; if it is not habitually well done, whatever edifice of culture may be built is warped and insecure. The conclusions which must be made by the score every day are of complicated nature; the facts are so many, so elaborate, so imperfectly known, that it is often very difficult to draw any conclusion, much less to be sure that we have the right one.

It is justly demanded that the school familiarize children with a mode of thought so ubiquitous, so important and so difficult. A subject suitable for this purpose should have three characteristics:

1. That its conclusions are *certain*. At first, at least, it is essential that the learner should *know* whether or not he has drawn the correct conclusion.

2. That it permit the learner to begin with simple and very easy conclusions, and to pass in well graded sequence to very difficult ones, as the earlier ones are mastered.

3. That the type of conclusions exemplified in the introductory subject be found in other subjects also, and in human intercourse in general.

These desiderata are possessed by mathematics in a far larger measure than by any other available subject.

1. *Certainty*. There *is* one, but *only one*, branch which may claim certainty; in which *experts* have not seriously disagreed, — Mathematics. In all other branches, experts, authorities (not merely beginners or bunglers), have disagreed, have often *contended*, and have not been able to convince each other, or the world at large. In mathematics there is substantial agreement. There are different schools of thought, but they are harmonious; supplementing, not antagonizing each other. Mathematics can decide whether or not its conclusions are right. The mathematician, more than any other scholar, can find his own errors or be convinced of them by others. This holds not only for the expert but down to the humblest beginner. There is no such thing as authority in mathematics. Nothing is accepted on the word of another; every one has a right to demand that *he himself* be convinced, and that the matter be not left unsettled. The conclusions of the tyro have as much weight as those of a congress of mathematicians, *i.e.*, each stands or falls according to its own demonstrated truth or falsity, quite independently of who is back of it. Mathematics possesses, then, the first desideratum, — *Certainty*; it is a subject (and the only subject) whose students are or can be *quite sure whether or not their conclusions are right.*[9]

The feeling of independent certainty should be reached early

[9] "Nor are the two elements, enthusiasm and drill, incompatible. Accuracy is essential to beauty. The very definition of the intellect is Aristotle's: 'That by which we know terms or boundaries.' Give a boy accurate perceptions. Teach him the difference between the similar and the same. Make him call things by their right names. Pardon him in no blunder. Then he will give you solid satisfaction as long as he lives. It is better to teach the child arithmetic and Latin grammar than rhetoric or moral philosophy, because they require exactitude of performance; it is made certain that the lesson is mastered, and that power of performance is worth more than knowledge. He can learn anything that is important to him now that the power to learn is secured; as mechanics say, when one has learned the use of tools, it is easy to work at a new craft." — Emerson, Lecture on Education.

by the pupil. Dogmatic statements are not to be accepted merely on the authority of the teacher or of a book. From the beginnings of arithmetic onward, it is possible so to present the subjects, that in the main the pupil sees a satisfactory reason for the processes employed, and does not unreasoningly accept mechanical rules on the teacher's authority.

Pedagogic Bearing.

2. *Simplicity.* Mathematics possesses also the second desideratum, — *Simplicity*. It begins, as is well known, with few and uncomplicated definitions and postulates, and proceeds, step by step, to quite elaborate cases. It would be difficult to find a subject in which a better gradation is possible, in which the work to be done can be better adapted to the powers and needs of the pupil at each stage of his advance than in mathematics. Its drill has been well compared to that of the athlete, who by carrying the same calf every day finally carried an ox.

3. *Applicability.* Mathematics possesses also the third desideratum, — *Applicability of the skill acquired*. The value of the study of mathematics has been more seriously questioned from this point of view than any other.[10] From the

[10] "An English naturalist, of most unusual merit, Professor Huxley, wrote recently: 'Mathematical instruction is almost purely deductive. The mathematician starts from a small number of simple propositions, the proof of which is so easy that they bear the name of *axioms*, and he has nothing further to do than to draw subtle conclusions from them. The teaching of languages, to the extent to which it is ordinarily carried, is of the same general nature. Authority and tradition furnish the *data*, and the mental operations are deductions. Mathematical science is a study which owes nothing to observation, nothing to experience, nothing to induction, nothing to causality.'

"Goethe has said:

"'Mistakes may be made by the wisest in the land
When they undertake to clear up what they do not understand.
(Verständige Leute kannst du irren sehen,
In Sachen nämlich die sie nicht verstehen).'

"This explains how a scientist so eminent as Huxley was able to make assertions so opposed to the most incontestable facts.

"In fact mathematical analysis constantly invokes the aid of earliest times down to the present century, the belief has been held that the conclusions of mathematics are so radically different from all others, that skill in making the conclusions of mathematics is of no help whatever beyond its confines.[11]

This opinion is to some extent still prevalent to-day, and it should be carefully considered. But before doing so we must

What is Mathematics?

first take up the question: "What really constitutes mathematics?" Different definitions have been given, varying somewhat in form, little in essence; that of Peirce[12] is clear, terse, and well fitted to be the basis of our considerations, viz.:

Mathematics is the Science of necessary Conclusions.

According to this definition those conclusions and no others are mathematical, which must be true, provided the premises are true. In mathematics, granted the premises, the conclusions follow inevitably; outside of mathematics, the granting of all the premises does not necessarily establish the conclusion.[13]

Thus, it is not a mathematical conclusion to say that the sun will rise to-morrow. For granting that the sun has risen ever

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new principles, new ideas, and new methods."—Sylvester, *Rev. de l'Instruction Belgique*, Vol. 12, p. 365.

[11] "Arithmetic and Algebra confine themselves to teaching in thousand fold ways identically the same theorems. The problems of life are more complicated: none is positive; none is absolute; one must guess, one must choose, and indeed with the assistance of observations and hypotheses which have no connection with the infallible course of calculation.

"Demonstrated truths emphatically do not lead to probable truths, and the latter only are met in business, in arts, in society. Nothing finds less application in life than a mathematical proof. A theorem in numbers is either true or false; in all other respects the true and the false are so commingled that often instinct alone can discern between different motives. . . ."—Mme. de Staël, *L'Allemagne*, Part I., Chapter 18.

[12] Peirce, *Am. Journal of Math.* IV.

[13] In this connection "The Fundamental Conceptions and Methods of Mathematics" by Bôcher (*Bull. Am. Math. Soc.* 1904, pp. 115–135) should be read; it contains a clear and instructive discussion of various definitions of the term *mathematics*.

so many times, and never failed once, it does not *necessarily* follow that the sun will rise to-morrow. It is conceivable that the sun may cease to exist before to-morrow. On the other hand, it is a mathematical conclusion to say that if:

All men are mortal,
And Socrates is a man,
Then Socrates is mortal.

Here the conclusion is a *necessary* consequence of the premises.

The familiar jargon of mathematics is missing, but the thinking is no less mathematical than:

All pentagons have five diagonals.
This figure is a pentagon,
Hence, this figure has five diagonals.

The fundamental question proposed above is: Does the study of mathematics, the science of necessary conclusions, help the mind to make better those conclusions which do not follow necessarily?

Necessary Conclusions from the Logical Point of View.

How is a conclusion made that does not follow necessarily from the known premises? Consider the conclusion that the sun will rise to-morrow. This does not follow from the premises; the sun might explode before to-morrow, or the earth might explode, or be brought to a standstill. The average man would consider these possibilities ludicrous; but they are not so. It may be, for example, that gases have been generating for centuries in the interior of the sun or of the earth, and that the increasing pressure will to-night reach the point of bursting the containing envelope. We simply know nothing whatever about such conditions. A being better informed than we might feel with as much conviction, and with stronger warrant, that the sun rose for the last time this morning, as we feel that it will rise again to-morrow. To us, however, the various contingencies just mentioned are quite intractable; so we eliminate them, and admit as working premises that the sun and earth will continue to exist in the same relative positions as heretofore, and that the earth will continue to revolve on its axis as heretofore. From these premises, together with the known facts, it is a mathematical consequence that the sun will rise to-morrow. The process consists of two steps: *first,* the substitution of tractable hypothetical conditions for the intractable real condition; *second,* a mathematical inference.

In this case the hypothetical conditions conform so closely to the real conditions as we know them, that the conclusion is accepted as certain. Even to indicate that the hypothetical conditions are not the real conditions would be regarded as ridiculous by many. They would consider anything other than the hypothetical conditions as practically out of the question.

A second example. Mr. A. agrees to meet me at a certain place and hour for a certain purpose. I go to that place because I believe that Mr. A. will also come there. How did I reach this belief? I know that Mr. A. has kept five previous appointments with me with exactitude; he has missed none. I disregard the thousand and one possibilities of accident that may prevent him from keeping this appointment, and accept instead as working basis that he will do in this instance as he did before, and from these premises I infer mathematically that Mr. A. will be there on time. As before, the process consists of two steps: *first,* replacing the real premises by hypothetical ones; *second,* making a mathematical inference from the hypothetical working premises.

This inference is accepted as the working conclusion, but I do not feel the same confidence in it as in the conclusion that the sun will rise to-morrow, because the difference between the working premises and the real premises seem to be much greater.

If, *as a third example,* I am thinking of making a business proposition to Mr. A., the conclusion as to what proposition to make him is reached in the same way, though very likely as the final result of a chain of intermediate conclusions.

In every case the conclusion is reached by the same two steps: the replacing of the actual but unmanageable conditions, by working hypothetical conditions, conforming as closely to them as possible; and the making of a mathematical inference from the working conclusions. The degree of confidence felt in the conclusion depends upon the divergence estimated to exist between the hypothetical and the real conditions. The estimation of this divergence is, of course, in itself a conclusion reached.

In every Conclusion, a Mathematical Inference.

It appears thus that in any reasoned conclusion the act of inference itself is always of a mathematical character; that is, it is the recognition of the necessary consequences of certain working premises. Of course the reasoning is usually not formal, and the steps are seldom even consciously recognized, but when all the tacit assumptions and inferences are clearly brought out, every reasoned conclusion will be seen to consist of the two parts named.[14]

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[14] This is in reality tantamount to claiming that all reasoning is *deductive*. Inductive reasoning can always be put into the deductive form, as was done above in the first example, the major premises asserting that whatever is true in the particular instances observed is always true. But deductive reasoning is often asserted to be sterile — in the sense that the conclusion must have been known before the premises could be asserted. This is true when the premises are truths which could only be established by examination of all the instances, as in the statement "All men are mortal"; but in mathematics the premises are very often purely hypothetical, or are simply definitions, and do not presuppose the existence of the objects themselves or examination of them if existent. Knowledge is increased by finding an actual concrete instance of the object. For example:

Every pentagon has five diagonals;
This figure is a pentagon;
Therefore, this figure has five diagonals.

The major premise is based on the definitions of "pentagon" and "diagonal." It does not presuppose the existence of any pentagon, much less the examination of all existent pentagons. By reasoning on the basis of the two definitions we feel certain that, whether or not any pentagon exists, every one that does exist has five diagonals. Granting all this, our knowlege is further increased when we find that *this* particular figure is a pentagon, and know, even without examination of it, that it has five diagonals. The

It is generally considered that experience and a wide knowledge of men and of the world usually lead to the formation of better conclusions. This simply means that the selection of the working premises and the estimate of the divergence between them and the real conditions approximates more closely to the mathematical type when made upon a wider basis of facts. Profiting by experience means progress towards the mathematical ideal. The general belief that one ought to profit by experience implies the conviction that as the basis of fact is widened, as the rôle of guessing, of instinct, is restricted, the conclusion should have a higher degree of certitude. The type of reasoning found in mathematics seems thus not only available but essentially interwoven with every inference in non-mathematical reasoning, being always used in one of its two steps; facility in making the other step, the more difficult one, must be attained through other than purely mathematical training.

The idea which has been discussed from the purely logical point of view also underlies all scientific researches and all reasoned every-day thinking about men or matter, viz., the assumption, usually unconscious but none the less real and essential, that every event, no matter how contingent in appearance, is in reality a necessary consequence of its antecedents; and hence, that it can be *deduced* from these antecedents by any mind which knows them all and is able to solve the complicated relationships by which they condition the event.[15] Contingency is

Necessary Conclusions from the Point of View of Every-day Assumptions.

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deductive reasoning of mathematics increases the individual's store of knowledge in two ways: *First*, it makes him know the consequences of certain definitions and assumptions, for example, that it follows from the definitions of pentagon and diagonal "that every pentagon has *five* diagonals." *Second*, it increases his knowledge of the properties of existing objects.

[15] See H. Spencer, *Principles of Psychology*, Part VI., Chaps. V., VI.

For an instance of use of mathematical procedure as model in speculations far removed from exactitude see H. Spencer, *Social Statics*, close of third section, "The Moral Sense Doctrine."

felt to be a consequence of inadequate knowledge of premises. Hence the constant effort in all domains of knowledge and all phases of social interrelations to increase the store of known facts and relationships. The conviction is, that by so doing the uncertainty of the conclusions is diminished. In practice an attempt is made to reach a rude approximation to the truth, by means of the known facts and working assumptions. The inferences from these premises must be acted upon, but with due regard to its approximate character. In all cases the mental activity is of the same type, differing in degree but not in essential character.

The mathematical inferences, simple and certain, are well fitted for first training for the making of the uncertain inferences of human life, but it is patent that mathematics is not a *sufficient* instrument to train to high skill in making contingent inferences. It is unequalled for the beginning of this training, on account of its simplicity and certainty, but it must be supplemented by other subjects, — languages, natural sciences, — in which by degrees the premises become less fully known, and hence the conclusions more uncertain, followed by those subjects in which the advent of the human will makes matters still more contingent, as history, psychology, the political and social sciences, and metaphysics.

Mathematical Methods outside of Mathematics.

When a mathematician points out looseness of reasoning, the objection is sometimes very specifically made that mathematical methods and ideals have no place outside of mathematics. "One cannot expect mathematical certainty." "You cannot carry your mathematical methods into non-mathematical things." This usually means a chafing against the recognition of the merely probable as such: he who thinks mathematically (this is not synonymous with mathematician) will not concede a greater degree of certainty than the premises warrant. He will act upon the probabilities, but nevertheless admit candidly that they are merely probabilities. The loose thinker is apt to raise the cry, "contingent reasoning is entirely different in kind from demonstrative reasoning," whenever his contingent conclusions are duly labelled, and not unhesitatingly accepted as assuredly correct. It would not be easy to find a better example of the existence of the mathematical spirit in non-mathematical affairs than the following sketch of the growth of a great group of industries. Note the mathematical march from premises to conclusion.

An Illustration.

"The large things began with an idea. Not considering the comparative advantages of New York and Chicago and Philadelphia, or of cotton, or shoes, or railroads, Mr. X. went back to first principles, went in search of 'power,'—water power,—some big, well-placed falls or rapids, which, if once penstocked and turbined, would draw manufacturers from everywhere into its plunging whorl. At S. he found it—horse power enough to grind the grist of half a continent. He built a 15,000 horse-power canal and awaited the manufacturers. They did not come.

"Then he sat still for a time and indulged in some theorizing. He had a cheap power, and the Great Lakes offered amazingly cheap transportation. If, then, there were available some *raw material* equally cheap, until the world should be surfeited with the product of that power there would be no limit to the amount of capital which could profitably be invested in that raw material and that power. In the wilderness to the north, 150,000 square miles practically unexplored, he found his raw material. While in Europe and the United States 'pulp' wood was constantly becoming scarcer, here were forests of spruce that he could not hope to exhaust in a thousand years, and that renewed themselves in thirty. There were no logging rivers, but a few score miles of railway would serve his purpose no less handily; so he proceeded to make pulp.

"There were dragons in the way, however. In the United States there was a hard and fast combination of paper makers who decided that it would be a wise thing to abstain from buying Canadian pulp until they could get it at their own price. In Canada there was practically no market, and since the pulp as then shipped was half water, the double weight made freighting to Europe not to be thought of.

"He decided that he must make *his* pulp dry. This inspiration aroused great hilarity among paper-machine men, who softened their hilarity only to show him how impracticable his ideas were. Then he decided that he would have to make that dry-pulp machine himself. He knew something about mechanics, and there were men obtainable who knew a great deal more. There were months of daily disappointment, but that machine was built and perfected. Mills were built whose daily output is now the greatest in the world. No big paper contract, even in Japan, is made without finding how prices will run at S.

"Mr. X. was already considering the money possibilities in treating the raw material with sulphite of lime. But they must have sulphur, which meant seeking prices of another combine, and that in Sicily. Now, to bring sulphur all the way from Sicily seemed unreasonable, in fact unnecessary. So he began to look nearer home for the yellow element. At the nickel mines he found it, but in combination with pyrrhotite ore, and the nickel men informed him that there was no way of separating them that would save it. He acknowledged that that was true by any method that was then in use. So he built a model laboratory, assembled about him practical and scientific men from all parts of the world, and their work was entirely successful. Then in the laboratory the question of utilizing the residue was coming up. The answer was breath-taking. By means of an original electric treatment the nickel and iron were fused into a nickel-steel alloy of the highest grade. Shown to the Krupps, they at once contracted for all that could be produced in the next five years.

"But again there seemed to be a large-sized 'fly in the ointment.' Much of the ore contained copper pyrites, and a very little copper in nickel-steel entirely ruins its efficacy. Once more, with that faith in modern science which is a kind of industrial religion with him, he had recourse to the laboratory. To remove the copper they needed caustic soda; this they obtained from common salt, which was within easy reach. Moreover, there was a ready sale for the by-products.

"But in the ferro-nickel the percentage of the latter metal was seven, double what armor-plate specifications call for. If, now, just a little more iron could be found. Taking once more to the wilderness, the Helen mine was discovered, a mountain of red hematite assaying more than sixty per cent pure. Now a steel man and ore boats were needed. As to the latter he was blocked. Another cheerful combination of his own countrymen informed him that there were none to be had. He promptly sent to England and brought back four ocean carriers, and a large part of their cargo consisted of Portland cement for the steel works, etc., etc."[16]

In this description the endeavor to exhibit the essentials, the central steps of the enterprise, leads to a chain of syllogisms, like those of mathematics in essence, differing only in subject matter, in degree of exactness, in complexity. The "if's" are analogous to those of mathematics, but the range of possibilities is much wider. When the manufacturer says, "If I had good water power I could do so and so," the attitude of mind is essentially of the same type as that of the mathematician who says, "If I could prove angle A equal to angle B, I could do so and so."

Contingent Conclusions in Mathematics.

What has been said relates to mathematics in its finished form. In the *study* of mathematics, if treated at all as it should be, contingent inferences must constantly be made. As soon as mathematics is taught as a subject in which the pupil thinks out results for himself, and does not simply assent to the deductively formulated results of others, much training in induction is given; if the concrete, experimental sides of mathematics and its close relation to the phemomena of nature are brought out, the drill in contingent reasoning is increased. In all such work, probabilities must be balanced, the advisability of trying this or that method must be judged, while at the same time this contingent work always carries with it the distinguishing advantages of mathematics; viz., simplicity together with ease and certainty in verifying whether or not the conclusion drawn was correct.

Summary.

To the objection that since mathematics has to do with necessary truth only, it gives no valuable aid in drawing contingent conclusions, we have made this reply:

1. Every contingent conclusion, if made best, must be made on the hypothesis that the facts known are correct and complete, and then make allowance for the *important fact* that they are not so. (The amount and nature of this

[16] Condensed from the *Saturday Evening Post*, Sept. 13, 1902.

allowance is again a contingent conclusion.) Contingent reasoning is but a broadened mathematical reasoning, and the latter may well begin the training for the former.

2. Easy contingent conclusions do occur in mathematics.

Another Objection considered.

It has been objected that whatever might be expected in theory, actual facts show that mathematicians are peculiarly unskilled in the contingent inferences of practical life, that they wish to treat all relations of life by formula.[17]

For the teacher and the professional mathematician, there is a real danger here. Since they busy themselves year in and

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[17] Napoleon, for example, appointed the great mathematician Laplace to an administrative position and was much disappointed with his work. "Laplace failed to grasp any question from its real point of view; he searched for subtleties everywhere, had only problematic ideas and carried the spirit of infinitesimals even into his administrative work." — Napoleon, cited in *Rebière, Mathématiques et Mathématiciens*, p. 185.

On the other hand, a large number of French mathematicians of the first rank have distinguished themselves as men of affairs. Thus, "Monge (1746–1818), author of two classical works, *Géométrie descriptive*, and *Applications de l'analyse a la Géométrie*, whose influence has been felt down to the present day, was Minister of the Navy in 1792; in the following years he accomplished incredible results in securing materials for the defence of the country, in 1794 he founded the *École polytechnique*, and in 1798 he accompanied Napoleon to Egypt. Carnot (1753–1823), Minister of War under the Convention, and later under Bonaparte, in the midst of his effective political activities wrote his well-known *Métaphysique du calcul infinitesimal* and his *Géométrie de position*. Joseph Fourier (1768–1830), the immortal creator of the *Théorie analytique de la chaleur*, displayed brilliant diplomatic ability as Commissioner in Egypt, suppressed with the greatest skill an insurrection of the inhabitants of Cairo, and with it all published a number of mathematical papers and collaborated zealously in the archæological *Description de l'Egypte*. In 1802 he led to a successful issue the previous attempts to drain the swamps of Bourgoin. Arago (1786–1853), prominent physicist and astronomer, was, as deputy under the Kingdom of July, the most feared speaker of the opposition. He was Minister of War and the Navy under the provisional government of 1848, and later year out with the easy and certain inferences of mathematics which demand only relatively simple contingent reasoning, it seems quite likely that when suddenly called upon without

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prominent member of the Executive Commission. Poncelet was lieutenant in Napoleon's Russian campaign, and while a Russian prisoner, deprived of all facilities for scientific work, he outlined his epoch-making work, *Traité des propriétés projectives des figures*, which secures to him, as the founder of projective geometry, a prominent place among the mathematicians of all times. On his return to France he re-entered the army and was made General in 1848. Freycinet, well known as Minister in our own times, is a mathematician of no little ability, and has published a number of noteworthy works."

"If in Germany the goddess *Justitia* had not the tiresome habit of always bestowing the ministerial portfolios in the cradles of her own offspring, who knows how many a German mathematician might not also have made an effective Minister?" — Pringsheim, *Jahresber. d. deutschen Math. Vereingung*, 1904, p. 371.

"Mathematicians fancy that their formulas are infallible because they are drawn from mathematics, and they have a formula for everything: everything is classed, ticketed, and in such a way as to preclude discussion. How can one dispute with a formula?" — Guyau, *Education and Heredity*. (Transl. Greenstreet.) New York, p. 245.

"To see these things (in study of nature) requires more than a mere mathematician; but the ablest mind which has never gone through a course of mathematics has small chance of ever perceiving them." — *Mill on Hamilton*, p. 537.

"Let us be assured that for the formulation of a well-trained intellect it is no slight recommendation of a study, that it is the means by which the mind is earliest and most easily brought to maintain within itself a standard of complete proof. A mind thus furnished and not duly instructed on other subjects may commit the error of expecting in all proof too close an adherence to the type to which it is familiar. That type may and ought to be widened by a greater variety of culture, but he who has never acquired it has no just sense of the difference between what is proved and what is not proved: the first foundation of the scientific habit of mind has not been laid. It has long been a complaint against mathematicians that they are hard to convince, but it is a far greater disqualification, both for philosophy and for the affairs of life, to be too easily convinced: to have too low a standard of proof. The only sound training or experience to decide some vastly more complicated question, they may be unable to do so well. It may be admitted without the slightest hesitation that the "*mere* mathematician," if such a *rara avis* could exist, would in all probability expect the reasoning of non-mathematical every-day life to conform to the simple standards of mathematics, and that his conclusions would often be woefully wide of the mark. But this is not the result of the study of mathematics but rather of the study of nothing else.

The danger in nowise threatens the secondary school pupil. The small modicum of mathematics which he receives, thoroughly intermixed with many other subjects, barely suffices to give him a glimmering of what constitutes a proof, to develop a little the power of careful inference, to give him a little insight into the mathematical march of nature, but need not arouse the slightest alarm lest he attempt to model his whole life on the deductive syllogism or the algebraic formula.

Even for the teacher the danger is not serious. Few are so immersed in their specialty, as not to have recreational interests along other lines. They simply share with the business man, the physician, the politician, the artisan, the limitation of knowing well one line only. That a mathematician cannot

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intellects are those which in the first instance set their standard of proof high. Practice on concrete affairs soon teaches them to make the necessary abatement; but they retain the consciousness, without which there is no sound practical reasoning, that in accepting inferior evidence because there is none better to be had, they do not by that acceptance raise it to completeness. They remain aware of what is wanting to it." — *Mill on Hamilton*, pp. 524–5.

"The 'Bugologist,' the man who knows nothing but what Latin names to attach to plants, animals, or stones, clouds, stars, etc., has long been the stalking horse or awful example of not only narrowness, but impracticability and ignorance of all that lies within the homely ken of common sense. Men who realize this ideal, have in my belief always been as rare as arithmetical or musical prodigies, who are also sometimes semi-idiotic. The danger is most felt by teachers of Grammar and High Schools and most often finds expression in their meetings." — G. Stanley Hall, *Confessions of a Psychologist*, Ped. Sem., 1901, p. 110.

usually handle the few business transactions that come in his way with the acumen of the experienced business man is no more to be wondered at than that this same business man cannot deliver a brilliant lecture on elliptic functions.

"No Head for Mathematics."

Another objection relates not to the subject matter, but to the pupil himself. It is claimed by pupils as well as others, that special mental qualifications, possessed by few, are requisite for the apprehension of mathematics. Those who are actually working in the field, who have had much experience in teaching mathematics to all types of pupils, have generally abandoned this opinion, so far at least as primary and secondary mathematics is concerned. The simple reasoning of school mathematics can be understood by any normal mind if properly presented. It is hard to see how any one really lacking such capacity could prove equal to the far more difficult reasoning demanded of him in any walk in life.

Are you to be a lawyer? How can you learn to analyse a complicated legal case, if you cannot learn to analyse a simple proposition of geometry? Are you a student of history? How can you determine the influence of Napoleon on the world's development, if you are incapable of determining the influence of a coefficient in a simple relation of algebra? Are you a linguist? How will you translate a masterpiece, with its myriad shades of meaning, from one language into another, if you cannot learn to translate a trifling "reading problem" into the corresponding mathematical symbols? Are you to be a physician? How will you diagnose and eliminate a disease, with its complicated, ambiguous, and obscure symptoms, if you lack the faculties needed to diagnose and eliminate the unknown quantity out of an elementary equation?

A Warning to Teachers of Mathematics.

But the teacher of mathematics may not ignore the fact that there are some among his pupils, their parents, and the general public who believe firmly that they are incapable of mastering mathematics. What is the basis of this erroneous belief? There are still even educators who hold the view that special qualities of mind are requisite for success in school mathematics. The fact that thoughtful men with excellent opportunities for observation can hold these views is the alarm signal which should lead teachers of mathematics to look sharply to their modes of instruction.[18] For the teachers of mathematics can by no means deal so leniently with himself as does Superintendent Knortz,[19] who attributes the fact that so many pupils have no interest in mathematics to an inherent lack of talent for it. Granting that the normal pupil *can* master school mathematics intelligently, the question comes home to us teachers of mathematics, "Why does n't he?" This belief and this question have controlled the preparation of the present work.

The Reasons advanced.

An examination of the reasons advanced for the belief that mathematical aptitudes are not common will be very instructive for the teacher. These objections in part confound school mathematics with professional mathematics, and raise the question, what in the teaching of school mathematics has given this impression to the lookers on? In part, these objections are directed against the tyranny of the abstract which is now also being combated in

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[18] "Poor teaching leads to the inevitable idea that the subject is only adapted to peculiar minds, when it is the one universal science and the one whose four ground-rules are taught us almost in infancy and reappear in the motions of the universe." — Safford, *Math. Teaching*, p. 19.

[19] "It is a well-known fact that relatively few pupils have real talent for mathematics, and consequently have no interest in it. 'How shall a deaf man fiddle?' says the proverb. He who is color blind does not become an artist, and he who has no feet cannot be a dancing-master." — Knortz, *Individualität*, Leipzig, 1897, p. 13.

This paper is by an American superintendent and from the American point of view, though written in German and published in Germany. If what is cited is interpreted in the light of the last sentences, one can well agree with it. It may readily be admitted that there may exist minds sufficiently crippled to have no capacity for mathematics, but experience indicates that the mathematically impotent mind is as rare as a color-blind eye or footless body.

mathematical circles;[20] in part, they hold the mind of the pupil responsible for faults of instruction.[21] Perhaps the belief is to be ascribed in large measure to too rapid and difficult work, especially at the outset. If there is sufficiently careful and easy work at the beginning, if the rate of progress is determined by whether or not the pupils have mastered what

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[20] "Another error of the curriculum is to suppose the mathematical mind much more common than it is, — not every one is a mathematician. There is in mathematics a powerful abstraction which repels most minds." — Bersot, *Questions d'enseignement*, Paris, 1880, p. 7.

"The question has been much discussed whether a certain measure of mathematics can be grasped by all pupils, and theoretically it is easy to reply, that mathematics is a science of pure reason, and whoever can think at all must be able to learn mathematics to the extent that it is taught in schools. But the facts do not corroborate this theory. Mathematics is not merely a matter of reason, but it requires also space imagination and grasp of abstract things, and a special type of memory which not all are likely to have in sufficient degree. Still less can every one apply the theorems and truths learned, and while not denying that with sufficient time and skilful instruction all pupils can be made to comprehend the theorems and proofs of school mathematics, still with regard to the power of imaging which is requisite for solid geometry and for the demonstration of original propositions, and most of all for descriptive geometry, a marked difference between pupils will always be noticeable." There follow a few suggestions for strengthening the space imagination by giving simple proofs without figure, and by making models. — Schiller, *Handbuch. d. Prakt Pædagogik*, Leipzig, 1894 (3d ed.), p. 638.

[21] "To ask whether a child has an aptitude for mathematics is equivalent to asking whether he has an aptitude for reading and writing." — Laisant, p. 187.

"It is not too much to say that nine-tenths of those who dislike arithmetic, or who at least feel that they have no aptitude for mathematics, owe this misfortune to wrong teaching at first: to a method which actually thwarts the natural movement of the mind, and substitutes for its spontaneous and free activity a forced and mechanical action accompanied with no vital interest, and leading neither to acquired knowledge nor developed power." — McLellan & Dewey, *Psychology of Number*, p. 146. See also Safford, *Mathematical Teaching*, p. 19.

is in hand, and not by an apportionment made in advance, and if the pace is set according to the needs of the slowest earnest student, rather than by the abilities of the quickest, pupils will not be muddled at the very outset, and will not give up the struggle in the settled belief that they are constitutionally lacking in a certain mathematical sense which many of their fellows seem to have.

Pedagogic Bearing.

The training which we have been discussing, and which is at the bottom the chief value and end of the study of secondary mathematics, can be obtained only in actually drawing the conclusions one's self, and not, to any noteworthy degree, in simply assenting to the correctness of conclusions drawn by some one else; *e. g.*, the writer of a text, or the instructor.

The analogy of a chess player [22] has been very aptly used to illustrate the various mental attitudes towards the acquisition of mathematical knowledge. One who simply reads understandingly a mathematical text, or hears understandingly a mathematical lecture or explanation, is like an onlooker at a game of chess, who sees that each move is made in accordance with the rules, and that finally one player or the other wins. It is a great step in advance when the player sees also *why* these moves were made rather than some others, that are also in accord with the rules. This likewise has its evident analogy in mathematics. But to get the real exhilaration of the game of chess one must play one's self. To watch others play, to analyze the motives of their moves, is essential to mastery of the game, but is effective only as it finds application in actual play. It is actual play, begun as soon as the moves of the pieces are known, that develops the need for theoretic study of the game. It would be a very tedious way of making a good chess player to require him simply to follow the play of others, and never to permit him to play himself.

The text and the teacher's suggestions should simply assist and guide in the work, and not actually do it. The cream of

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[22] Poincaré, *Report Paris Math. Cong.*, 1900, p. 125.

mathematics consists of the so-called "originals."[23] Whenever the student draws the conclusions himself, does his own work, he is doing *original* work, whether this is formally stated or not. Without work of this character mathematical study is almost useless for culture. (See chapter Heuristic Method.) In such work, correctness and certainty are the first desiderata. The pupil should therefore not be hurried in any way, but given all the time he needs. Not *how soon*, but *how well*, is the question.

Memorizing. Some pupils are tempted to evade precisely that portion of the work which gives the benefit, by memorizing the results of the work of others. This temptation is great to some pupils, and perhaps no other subject can become so barren and dreary as mathematics so studied. Ten pages of mathematics *understood* are better than a hundred memorized and not understood, and one page actually worked out independently is better than ten pages clearly but passively understood. The question is not *how much?* but *how?* The object is mastery, attainment of the spirit of the subject, and not to train the memory, or to ingest a large bulk of mathematical fact and formulas.

Sir Wm. Hamilton on the Study of Mathematics. Before leaving this subject mention must be made of the most elaborate and able attack on the study of mathematics that has ever been made. In the *Edinburgh Review* (1836) the eminent Scotch philosopher, Sir William Hamilton, paid his respects to mathematics as an exercise for the mind. It is not necessary to rehearse his statements in detail, or to attempt to refute his arguments.[24] They are by no means unassailable, still we may admit every one of his real arguments and remain as firmly convinced of the study and value of mathematics as ever. Two of his principal contentions are:

[23] "Truth must be ground for every man by himself out of its husk, with such help as he can get, indeed, but not without stern labor of his own." — Ruskin, *On Sheepfolds*, p. 22.

[24] See the excellent reply by Stuart Mill in his criticism of Hamilton's philosophy.

1. *The excessive study of mathematics is deleterious to the mind.* Mathematics ought, therefore, not to constitute or even to dominate university instruction.

We have already discussed this point. In secondary instruction to-day, mathematics holds a sufficiently minor place to satisfy even Hamilton.

2. *The study of mathematics is so easy that it affords no real discipline.* Discipline presupposes exertion; mathematics requires no exertion; all its conclusions can be apprehended without effort, since they follow necessarily from the premises.

There is, no doubt, truth in this assertion. Hamilton directs himself exclusively to the comprehension of mathematics when presented in a finished syllogistic form. If such mathematics is not forced on a mind prematurely, its difficulties are of a low order, — difficulties of language (understanding the words and symbols) and of memory. The benefits are correspondingly small. But mathematics is no longer taught as a purely passive subject to-day. The twentieth-century students of the pedagogy of mathematics would be the first to agree with Hamilton as to the small value of the subject when taught as he presupposed.

The reading of his paper is instructive to the teacher of mathematics, as are all thoughtful judgments upon the subject or any of its phases, from those who look at it from a different viewpoint. It is a pity more such criticisms are not made. Carefully studied, Sir William Hamilton's paper shows that much progress has been made in the pedagogy of mathematics since his time, and should serve to spur on or to encourage the teacher in his efforts, often enough quite arduous, to keep the pupil at work actively evolving his own mathematics.

We may now return briefly to Huxley's assertion (p. 21) that "mathematical science is a study which owes nothing to observation, nothing to experience, nothing to induction, nothing to causality."

Pupils who are taught in the belief that the chief ends of the teaching of elementary mathematics are those set forth above, will continually make observations and inductions. The observations will largely be of mathematical phenomena and relationships, and the inductions will be made from these; but they are none the less observations and inductions. Of late years, moreover, there has been growing emphasis on the experimental side of mathematics, and on the origin of the notions and problems of mathematics in the concrete facts of experience. This tendency will be discussed more fully later. At present it may suffice to say that observation, experience, induction, are essential elements of much of the best mathematical teaching of to-day.

Observation in Mathematics.

A Practical Objection.

Throughout this discussion the simplicity and certitude of mathematics have been emphasized again and again. The pupil may say, "I am not sure of my results; my thinking often proves not to be accurate; I find no exhilaration in the study of mathematics; quite the reverse." This may frequently be true. Many circumstances operate to hinder the easy and full grasp of mathematics in its simplicity and certitude, not the least of which is the impossibility of adapting the instruction fully to the needs of every individual, and the serious difficulties which even a slight overtaxing of the powers of the individual can cause. But when any considerable proportion of the earnest pupils of a class are floundering along in an uncertainty verging on despair, the caution signal is flying clearly for the teacher. It behooves him, believing in the simplicity and certitude of mathematics himself, to ask himself whether he cannot, despite the restrictions consequent upon teaching many pupils at a time, make some modification in his mode of instruction which will help the majority, if not all, of the struggling pupils into the land of mathematical freedom. This is one of the chief problems, yes, *the chief problem*, confronting the thoughtful student of the teaching of mathematics. For mathematics *properly* studied tends to strengthen, *does* strengthen, the power of thinking independently and accurately. The effect is doubtless greater than the pupil may himself perceive. In proportion as the effect is apparently small, so is the need great that the mathematical training be prolonged.

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Other Functions of Mathematics

The chief functions of mathematics in secondary education have now been discussed. Some other points of value may be added which are of considerable importance, though overshadowed by those which precede. **Other Values.**

1. *Generalizing conceptions; combining results.* Mathematics gives exercise in widening and generalizing conceptions, in combining various results under one head, in making schematic arrangements and classifications. It is easy to find instances of successive generalizations in the elementary field. Perhaps the most striking is that of the number concept itself, enlarged from that of the whole number, to include successively fractional numbers, irrational numbers, negative numbers, and imaginary numbers. One of the important aspects of algebra is its generalized treatment of the processes of arithmetic. In geometry, also, there are repeated occasions for grouping and generalizing results. Frequent synopses by the pupils, schematic arrangements of results relative to some topic, will help to develop the power to generalize and to classify.

Bain's assertion that mathematics does not teach how to generalize,[25] is, like much of Hamilton's paper, founded on the assumption that the pupil simply learns his mathematics ready made. When the pupil evolves his own mathematics, definitions and axioms included, the assertion is no longer tenable. As usual, the generalizations and classifications of mathematics are very simple and obvious in comparison with those of other domains of thought and activity.

2. *In formation and use of a symbolic language.* The symbols of mathematics constitute a language which is gradually developed by and for the pupil. This language must be acquired just like any other language, and there is need for

[25] "Mathematics does not teach us how to observe, how to generalize, how to classify. It does not teach us the prime art of defining by the examination of particular things." — Bain, *Education as a Science.*

genuine translation back and forth between the symbolic language of mathematics (a most effective shorthand) and the more prolix mother tongue. Long training with very gradual progress is needed to make the pupil feel at home in this language, and he has constantly to struggle not only with the difficulty of grasping the thoughts of mathematics, but also, often chiefly, with the symbolic form in which they are expressed. The training which mathematics gives in working with symbols is an excellent preparation for other sciences; much of the world's business, likewise, is accomplished by means of symbols. From the girl at the telephone board and the man in the signal tower to the president of a railroad or other great corporation, sitting in a small office and by means of symbols directing the myriad activities of a gigantic industry, the world's work requires constant mastery of symbols. Only those in the least remunerative and least desired occupations work *entirely* with the actual things, and with the advance to more responsible and desirable positions the work is done more and more largely by symbols. The drayman who brings goods to a store works almost exclusively with real things, but the proprietor handles little but symbols. In this twentieth century ability to work with symbols is indispensable to even the most moderate measure of success. The more important the position, the less closely the work is associated with the actual things affected, the more thoroughly symbolic it becomes, and everywhere, just as in mathematics, it is requisite both to be able to work with the symbols and also always to know what the symbols stand for. To be a mere juggler of meaningless symbols is fatal.

3. *In the scientific development of subjects to a finished form; the type form* (*ideal*) *for other sciences.* The rounded, finished treatment of a topic, the enumeration and consideration of what are evidently *all* the cases, is possible earlier and with more ease in mathematics than elsewhere. It sets the ideal towards which other knowledge strives.

4. *In early discoveries.* Mathematics gives an easy and early opportunity to make independent discoveries. The first

discoveries of all which the child makes are no doubt merely observational, but a little later, when his power of abstraction has been sufficiently developed to enable him to appreciate the greater certitude of mathematical conclusions, he can readily be brought to feel the exhilaration of genuine discovery in the field of mathematics, even when in other subjects the contingencies to be considered are still so numerous and elusive that he cannot feel that he has really grasped them all.

5. *As knowledge for its own sake.* When once the delights of independent mental activity have been tasted the pupil has had his first experience of what is really the *ideal* of all human effort, the exertion of mental powers on their own account, for the mere pleasure of doing. Holidays, recreations, mean not less work, but financially unproductive work, — work that is undertaken not as a means of livelihood, or under stress of set hours and tasks, but just as the impulse may move us. Play has well been called "unremunerated work." The ideals, the castles in the air, the grand goal of men's strivings, all reduce to some variety of exertion for its own sake. In the intellectual life, no draught of this great elixir of all men's hopes can be given at once so early and so satisfying as from the cup of mathematics. And let the teacher remember that he must and can be the cup-bearer.

It is all the more important to lay stress on knowledge and the exertion of intellectual power for its own sake as a sufficient end of education in view of the evident fact that the more general the diffusion of education, the smaller will be the utilitarian advantage of being educated. This advantage is relative. When all people are high-school graduates, the mere possession of a high-school diploma will not have the slightest utilitarian value. In various European countries there may be noted the bad effects of greater diffusion of secondary education, coupled with the expectation that the utilitarian advantages will not thereby be diminished.

6. *As cultivating reverence for truth.* Mathematics insists on the true, without regard to authority, tradition, self-interest, or prejudice.

"As pure truth is the polar star of our science, so it is the great advantage of our science over others that it awakens more easily the love of truth in our pupils. Languages also have their necessary formulæ, but they do not come to conscious recognition by the pupil who must learn a dozen exceptions to every rule. If Hegel justly says, 'Whoever does not know the works of the ancients, has lived without knowing *Beauty*,' Schellbach responds with equal right, 'Who does not know mathematics, and the results of recent scientific investigation, dies without knowing *truth*.'"[26]

Reverence for truth can be cultivated only if the teacher observes scrupulous honesty towards his pupils. He need never hesitate to use theorems without proof, to give "proofs" avowedly only plausible, to admit that exceptions to the validity of a theorem may arise, to acknowledge that he does not know; but let him never palm off the imperfect as perfect.

7. *As cultivating the habit of self-scrutiny.* Nowhere is the demand for rigorous scrutiny of one's own work so inexorable as in mathematics, and nowhere else does even slight delinquency so readily and unmistakably betray itself. This self-scrutiny can also be fertile as nowhere else, that is, it can give that well-grounded self-confidence, that well-assured encouragement to proceed which is so essential to the beginner. To attain this, however, the pupil must be held to check his own work and to assume the responsibility for its correctness.

8. *On the esthetic side.* Mathematics has beauties of its own — a symmetry and proportion in its results, a lack of superfluity, an exact adaptation of means to ends, which is exceedingly remarkable and to be found elsewhere only in works of the greatest beauty. It was a felicitous expression of Goethe's to call a noble cathedral "frozen music," but it might even better be called "petrified mathematics." The beauties of mathematics — of simplicity, of symmetry, of compactness, of completeness — can and should be exemplified even to young children. When this subject is properly and concretely presented, the mental emotion should be that of

[26] Simon, *Mathematischer Unterricht*, p. 21.

enjoyment of beauty, not that of repulsion from the ugly and unpleasant.[27]

9. *In the development of the imagination.* Mathematics makes constant demands upon the imagination, calls for picturing in space (of one, two, three dimensions), and no considerable success can be attained without a growing ability to imagine all the various possibilities of a given case, and to make them defile before the mind's eye.[28]

10. *In cultivating the power of attention.* There is no need to dwell upon this point. Attention is requisite everywhere, but in mathematics the slightest inattention is fatal and at once betrays itself. The pupil himself feels the need of unflagging attention and the impossibility of success if the wits are allowed to go wool-gathering.

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[27] "No doubt you have often been asked what mathematics is good for, and if those delicate constructions which we draw wholly from our mind are not artificial and children of our caprice.

"Among the persons who put this question I must make a distinction. Practical people ask of us only instrumentalities for making money. They do not deserve an answer; they should rather be asked what is the good of accumulating so much wealth, if, to have time for gaining it, they must neglect art and science, which alone give us souls capable of enjoying wealth. '*Et propter vitam vivendi perdere causas.*'

"Mathematics has a triple end. It is to furnish an instrument for the study of nature. But that is not all. It has a philosophic end, and, I dare say it, an esthetic end. . . . Those skilled in mathematics find in it pleasures akin to those which painting and music give. They admire the delicate harmony of numbers and of forms; they marvel when a new discovery opens an unexpected perspective; and is this pleasure not esthetic, even though the senses have no part in it?" — Poincaré, *Sur les rapports de l'analyse pur et de la physique mathématique.* Report Internat. Cong. Math., Zurich, 1897, p. 82.

[28] "It is as great a mistake to maintain that a high development of the imagination is not essential to progress in mathematical studies as to hold with Ruskin and others that science and poetry are antagonistic pursuits." — Hoffman, *Sphere of Science*, p. 107, 1898. See also Sylvester, *Nature*, I., p. 238; Carpenter, *Mental Physiology*, London, 1881, pp. 503–8.

11. *In fostering habits of neatness and accuracy.* The value of mathematics in this respect need not be urged, but only that the instruction should sedulously aim to attain these ends. The best results can be achieved only by constant vigilance and insistence on neatness and accuracy from the very earliest years.

Conclusion

Summary of the Discussion. The values of the study of mathematics that have been discussed may be summarized under the following heads:

1. *Its utilitarian values.* For direct practical usefulness, mathematics was seen to be second only to the mother tongue.

2. *As a fundamental type of thought.* Mathematics is one of the few characteristic types of human thought; no civilization has ever failed to evolve it, and with essentially the same results.

3. *As a tool for the study of nature.* — While mathematics as a type of thought seems to inhere in the human mind, the study of nature most frequently stimulates the mind to the mathematical type of thought, especially in the earlier stages of mathematical development, and the phenomena of nature cannot be thoroughly understood without mathematics.

4. *As exemplifying especially well certain important modes of thought.* The modes of thought explicitly mentioned were: understanding statements, noting facts, and making inferences. The unique value of mathematics as a first introduction to these arts lay in the certainty, simplicity, and possible gradation of its inferences. The skill gained from the study of mathematics was also found to be available beyond the bounds of mathematics, whose form of reasoning is the ideal towards which all other reasoning strives; owing, however, to the simplicity and narrow range of mathematical inferences, mathematics can only give the beginnings of the requisite practice.

5. *Other values.* Several minor points of value were also mentioned, in some of which mathematics has characteristic or special value, in others of which its value, though real and important, is in no sense unique.

The purpose of the discussion as set forth at the outset may be recalled in closing. The points of value of the study of mathematics have not been enumerated for controversial purposes, though objections have been considered, but because a study of the problems and methods of the teaching of mathematics can be well grounded only if based upon a careful study of the ends to be achieved by the instruction. These ends must be consciously planned for and worked for by the teacher, from whose point of view the benefits derived by the pupil would be premeditated results, and not accidental good fortune.

Official Statements of Aim of Mathematical Instruction.

It may be of interest to cite the statements of the aim of instruction in mathematics which are part of the official formulations of the curricula of certain countries, for these statements are thus formally promulgated with the purpose of fixing the standard of instruction throughout countries in which they are authoritative.

"For the secondary schools, the most important task of instruction in mathematics lies in a training of the mind which enables the pupil to use correctly in his own independent work the intuitions and knowledge which he has acquired. In all domains of this subject the aim must therefore be to attain a clear understanding of the theorems to be developed and their deduction, as well as practice and skill in their use. Like every other subject, mathematics also must pay special attention to the due cultivation of the mother tongue." — *Prussian Curricula of 1901.*

"Instruction in mathematics has in general the important duty of co-operating in the development of the power of thought of the pupils, to lead them to the formation of independent judgments, to facilitate the understanding of the laws of nature, and no less than any other branch of instruction to cultivate the clear expression of thoughts in correct language.

"Consequently, such portions of elementary mathematics have been incorporated in the curriculum as have a recognized high culture value, and in an order corresponding to the progressive mental development and power of comprehension of the pupil. Incidentally, the selection also had to take into account the needs of practical life and the connection of mathematics with other fields of knowledge, notably with the natural sciences." — *Austrian Curricula of 1900.*

To these may be added the following from the important report of the commission appointed in 1904 by the *Society of German Natural Scientists and Physicians* to investigate questions relative to instruction in mathematics and the natural sciences.[29]

"With full recognition of the formal culture value of mathematics, one-sided and practically meaningless special topics may be omitted, but on the other hand the power of viewing mathematically the world of phenomena surrounding us should be developed as highly as possible. From this there arise two special problems: the strengthening of the power of space intuition and the training to habit of functional thinking. The task of logical training, from time immemorial allotted to mathematics, is not hampered thereby, but we can say that this task only gains through the more pronounced fostering of the aspects mentioned, for thereby mathematics is brought into closer relation with the other domains of interest of the pupil, in which he is to set his logical powers to work."

The concluding aim for the highest class (corresponding roughly to the American Freshman class in college) is threefold:

"1. A scientific survey of the organization of the mathematical subject matter previously treated.

"2. A certain power of mathematical interpretation and its use in the treatment of problems.

"3. Finally and chiefly, insight into the importance of mathematics in the exact knowledge of nature." — *Report*, Leipzig, 1905.

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[29] For fuller account of the history and work of this commission see Young, *Bull. Am. Math. Soc.* April, 1906; Young, *Science*, May 18, 1906.

The German Society for the Advancement of Instruction in Mathematics recently formulated the purposes of instruction in mathematics in secondary schools as follows:

"In the secondary schools mathematics should be a part of general culture and not contributory to technical training of any sort; it should cultivate space intuition, logical thinking, the power to rephrase in clear language thoughts recognized as correct, and ethical and æsthetic effects; so treated, mathematics is a quite indispensable factor of a general education in so far as the latter shows its traces in comprehension of the development of civilization and the ability to participate in the further tasks of civilization. Accordingly applications of mathematics to problems from the field of the natural sciences, geography, and the relations of human society are to be constantly made, though without endangering the independent importance of mathematics." — *Unterrichtsblätter f. Math. u. Naturwiss.* 1904, p. 128.

Time and Scope of the Study of Mathematics

When should the Study of Mathematics be begun?

Harris [80] has pointed out that in the first stages of the development of the mind, the mathematical process is decidedly more complex than the other mental processes which are taking place at that time.

"The reason why it requires a higher activity of thought to think quantity and understand mathematics than it does to perceive quality (or things and environments) lies right in this point. The thought of quantity is a double thought. It first thinks quality and then negates it, or thinks it away. In other words, it abstracts from quality. It first thinks thing and environment (quality), and then thinks both as the same in kind or as repetitions of the same. A thing becomes a unit when it is repeated so that it is within an environment of duplicates itself."

Several very important consequences for the practical teaching of mathematics can be drawn from the fact formulated.

1. The mathematical process may not be introduced before

[80] *Psychological Foundations of Education*, p. 343.

there is a considerable stock of qualitative facts in the child's mind on which to work, and not until the child's mental powers are sufficiently developed to take the steps implied in even the simplest mathematical concept. It is a question whether we are not tending to introduce the abstractions of mathematics too early. The German boy who enters the gymnasium at the age of *nine* is expected to know only the four fundamental operations on integers, and in his first year (corresponding to our fourth grade) he learns further only the German weights and measures (decimal system) and the simplest operations with decimals; [81] by this time our children are introduced to the complexities of fractions, common and decimal, to our system of weights and measures, far more complicated than the international (decimal) system used in Germany, and even sometimes to percentage and some use of generalized (literal) numbers.[82] And yet the German boy does not come out behind at the end of the race ten years later.[83]

2. When introduced the quantitative concept should be developed as an outgrowth (abstraction) from the qualitative experiences. The two steps should be taken one at a time as far as is possible.

3. Repetition of the process is necessary. It is a mistake to assume that the quantitative concept once obtained (*i. e.*, the process of abstraction once intelligently followed by the child mind) is a permanent possession. The work should remain in constant touch with the qualitative, and the step from the qualitative to the quantitative should be made over and over again.

4. With due modification as to subject matter, what has been said is in essence applicable throughout the entire course to the close of the secondary school and even beyond.

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[81] Young, *Teaching of Mathematics in Prussia*, pp. 33, 47.

[82] See various text-books.

[83] Young, *l. c.*, p. 106.

To what Extent should the Study of Mathematics be carried?

It has been pointed out in what precedes that mathematics is particularly adapted for the beginning of training in making inferences; that the complexity of mathematics can be made to keep pace with growth of the reasoning power, and also that less exact subjects should be taught simultaneously.

When should the simplest subject, mathematics, be laid aside, and all the time given to the others?

From the cultural point of view, whenever mathematics has done its work. When the tasks of mathematics can be accomplished with facility, it is *prima facie* evidence that the powers which mathematics is intended to develop, are already developed. The pupils who are "good in mathematics" have from this point of view the least need to study mathematics, but those who do not succeed with mathematics are by that very fact notified that they ought to continue the study; not by proceeding to more complicated mathematics, but by trying again to master that which they have already attempted. It would be foolish for one who cannot master the simple relations of mathematics to think of devoting himself exclusively to the more complex subjects.

As types of thought, every pupil should have, in the secondary school, thorough courses in arithmetic, geometry, algebra, with their differing methods; if he goes to college, he should be introduced to the powerful and more modern methods of analytic geometry and of the calculus. If he succeeds well in grasping the spirit and solving the problems of all these diverse and characteristic forms of mathematical thinking, he may perhaps feel satisfied that he has sufficient acquaintance with typical phases of mathematical thought, so far as the demands of general education are concerned; but lack of success in any of these subjects points out a weak point which should not be left unstrengthened.[84]

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[84] "But the peculiar culture effect of this science (mathematics) can only be secured by use of its own peculiar material, and hence it is necessary that the mathematical subject matter be taught to a

All the subjects named also seem desirable as to content: in part as directly useful, in part as interesting from their close connection with the phenomena of nature and the material activities of mankind. For students continuing their work long enough, it is highly desirable to round off the work in mathematics with an introduction to the calculus, which is in so important a degree the mathematical study of the laws of nature.

It may be remarked in this connection that the recent changes in the programmes for the French secondary schools (Lycées) have made the elementary notions of analytic geometry and of the calculus a part of the work in mathematics required of all pupils, and there is a strong feeling in Germany in favor of including some calculus in the minimum mathematical requirement. The tendency of the movements for improvement of mathematical teaching in England and America is also to push back the introduction of these notions to an earlier stage.[35]

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certain extent, in order that it may exert its real influence. This consists of thorough culture in logic development of space intuition, skill in computation, and in the understanding of representation of space objects in the plane, and finally in the ability to understand in later life popular addresses and papers of a scientific character." — Schiller, *Prakt. Pädagogik*, p. 637.

[35] "In agreement with Perry it would seem possible that the student in the secondary school might be brought into vital relation with the fundamental elements of trigonometry, analytic geometry, and the calculus, on condition that the whole treatment in its origin is, and in its development remains, closely associated with thoroughly concrete phenomena." — Moore, *Presidential Address.*

CHAPTER III

METHODS AND MODES.

In the study of the pedagogy of mathematics, the point of view is sometimes that of the manner in which the subject matter is arranged and developed; at others that of the manner in which it is presented to the pupils. To introduce this distinction into the nomenclature, the former has sometimes been called *method* and the latter *mode*. In this usage, one would speak of the analytic *method*, but of the recitation *mode*. The distinction is not always easy to make; not all processes of instruction can be readily classified as relating distinctly and exclusively either to the sequence and interrelation of the subject matter, or to the devices by which it is made clear to the pupil. Nevertheless, the broad distinction exists, and even though the term "method" has been used to denote both phases indiscriminately, it may help to keep the distinction more explicitly in mind to use the terms, in the present chapter at least, loosely in the sense cited.

Methods and Modes: the Distinction.

Methods in Mathematics

As leading methods in mathematics may be mentioned: the synthetic, the analytic, the deductive, the inductive, the socratic, the heuristic, the laboratory. The characteristics of these methods will be indicated briefly in the sequel. They are not mutually exclusive; they shade into each other, and the classification of the treatment of a subject, topic, or problem under one or another method is often difficult. But though classification is sometimes hard or even impossible, the classes exist, and in their most typical and pronounced forms each of the methods has its marked characteristics and its peculiar adaptation to special situations.

Methods to be considered.

The Synthetic and the Analytic Method.

The *synthetic* and the *analytic method* are so familiar that their characteristics need only be recalled by a word. The synthetic proceeds from the known to the unknown; the analytic traces out a path from the unknown to the known. The synthetic says, "Since A is true, it follows that B is true"; the analytic says, "To prove that B is true, it is sufficient to prove that A is true." The synthetic "puts together" known truths, and by the combination perceives a truth theretofore unknown; the analytic "pulls apart" the statement under question into simpler statements whose truth or falsity is more easily determined.

Examples.

The usual form of statements of proofs in text-books of elementary geometry is a good example of the synthetic method. Beginning with known definitions and assumptions (axioms), each proof, each step, is deduced from what is known.

The solution of a quadratic equation may be taken as a specimen of analytic procedure.

The problem is to find for what value or values of x, if any,

$$x^2 + px = q.$$

The problem is solved if the same problem is solved for

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4},$$

$$\text{or for} \quad \left(x + \frac{p}{2}\right)^2 = q + \frac{p^2}{4},$$

$$\text{or for} \quad x + \frac{p}{2} = \pm\sqrt{q + \frac{p^2}{4}}.$$

The last is solved if *both*

$$x + \frac{p}{2} = +\sqrt{q + \frac{p^2}{4}} \quad \text{and} \quad x + \frac{p}{2} = -\sqrt{q + \frac{p^2}{4}}$$

are solved.

Examples of geometric analysis will be given in the chapter on the teaching of geometry. In a previous chapter, an example of analytic reasoning outside of the domain of mathematics has already been given (pp. 28–30).

The analytic method is the method of the mathematical worker, the synthetic method is that in which he usually presents his results. That the analytic method should in general be that of the class-room admits therefore of little doubt. Each step in an analytic march has its reason, its purpose. In the synthetic method the steps follow more or less blindly; the truth of each is evident, but why this step should be taken rather than some other is a mystery, and the final result is often reached with a disagreeable shock. "How did the author find this proof?" is frequently asked by pupils. The reply is that in all probability he found it in an entirely different way from that in which he presented it to the world. Not one proof out of a hundred is found by synthetic steps, and many synthetic forms of statement bear little trace of the analytic path of their discovery, so that the pupil justly has the feeling of one led about blindfold. The synthetic method *proves*, but often does not *explain*.

The Function of these Methods.

The great advantage of the analytic method is that if it connects with the known at any point, no matter where, its task is achieved; the synthetic method, on the other hand, has only a single point to reach. The synthetic method seeks "a needle in a haystack"; in the analytic method the needle seeks to get out of the haystack.

The synthetic method is that of logical exposition; it will usually succeed most rapidly in producing the conviction that particular statements are true, but it does this at the price of the minimum of intellectual benefit to the learner. Since the attitude of the pupil should in general be active, not passive, that of the discoverer, not that of the learner, the mathematical subject matter should usually come to him in analytic form.

For permanent record, in printed books, for example, the synthetic form has the advantage of being more finished, more certain, more formal, while the analytic method is informal, tentative, and, when reduced to cold black and white, may even seem colloquial. In how far text-books should be written analytically is an open question, but there is no question that the atmosphere of the mathematical class-room should be analytic from the primary school to the University.

The Synthetic Method in the Class-room.

Has, then, the synthetic method no place in the class-room? It has, and a most important place. It is the method in which, in the class-room as well as in publications, the discoveries made analytically may usually be best arranged and surveyed. The synthetic presentation shows the unfaltering, sure-footed march of mathematical demonstration from the known to the unknown, and a demonstration reached after much analytic groping, with many "if only's" and "how's," should at once be cast into permanent shape in the synthetic mold.

For formal statements of proofs obtained, for their permanent record, for summaries, for reviews, the synthetic method is invaluable in the class-room.

Geometric Analysis.

What has been said applies equally to the geometric and the algebraic side of mathematics. The terms *Geometry* and *Analysis* are sometimes used in contrast, and the usage is no doubt due to the fact that analytic forms of presentation were used in the algebraic field earlier than in geometry. But there is *geometric analysis* as well as algebraic analysis, and the demonstrations of algebra and of analytic geometry may also be cast in synthetic form.

The Deductive and the Inductive Method.

A word will serve to recall the character of the *deductive* and the *inductive method.* The deductive method proceeds from the general to the particular; the inductive, from the particular to the general. A typical deductive syllogism is:

> All men are mortal.
> Socrates is a man.
> Therefore, Socrates is mortal.

A typical inductive inference is:

> The sun has risen each past day of which we have any knowledge: therefore the sun rises every day.

The deductive type of inference is precisely what has been defined in the previous chapter as the *mathematical* type. It is the final form of all mathematical reasoning, but it does not follow that the reasoning which leads to the result is entirely or even in part of this type. On the contrary, it is usually largely inductive. "This problem seems like such and such that I have met before; I solved them in a certain way. Therefore I can solve the present problem in the same way."

Mathematics in the synthetic finished form is deductive; mathematics in the making is inductive. Not only is the plan for the work inductive, but the theorems or processes themselves are very often discovered inductively, by the consideration of special examples. For the learner, the inductive method of approach is as a rule decidedly the best. By the consideration of quite a number of special instances he begins to see some general theorem or property underlying them all, and is thus led to try to find a deductive proof of the truth of the theorem or the existence of the property.

Mathematical Discovery Inductive.

The belief that the theorem holds was reached by real induction and a purely inductive science would be obliged to leave it thus, but it is one of the chief glories of mathematics that it can take its theorems from the realm of inductive probability into that of deductive certainty. The question of whether or not an inductive inference is correct is one that need not be left unsettled in mathematics.

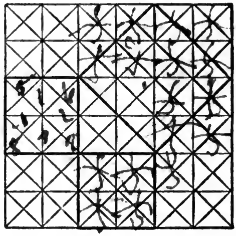
These considerations have important bearing upon the work of the class-room. Even in mathematics, which far more than all other sciences is regarded as a deductive science, induction must have a prominent part. The teacher cannot study too carefully the rôles that inductive and deductive reasoning play in mathematics, but it need hardly be said that the pupil would profit little by any formal discussion of these methods. His attention should be confined to the actual reasoning, and not diverted to any more or less introspective discussion of the character of the reasoning. As to the work the pupil is asked to do, the opinion is widely held that inductive work should be given a more prominent part in the class-room work. It is now extensively believed that it is not best to announce a theorem, then give a strict

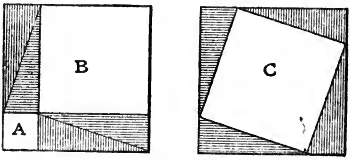
Induction in the Class-room.

deductive proof of it, and finally, perhaps, apply it in some problems. The more modern method would be: First, to give the pupil some specific problems, as practical as possible, foreshadowing or leading up to the theorem in question, this to be continued until the pupil himself (with some prompting, if necessary) announces the theorem and sees the need for its rigorous proof. He is now ready for this proof, and after it is given, more applications of it should follow.

An Example.

The theorem of Pythagoras, for example, would be begun by telling the pupil to draw a tessellated pavement (see figure), and by counting small triangles to compare the areas of the squares marked more heavily. Then various right triangles with integral sides might be constructed and the areas of the squares compared by measurement, and (in a few cases) by weighing. The pupil will thus be led to announce the Pythagorean theorem





himself and will welcome a proof of it, or hints enabling him to devise a proof. The proof might well be in the first instance a "dissection proof," which the pupil would actually cut out himself; for example, that indicated in the figure.

The *Socratic method*[1] consists in securing the pupil's assent

[1] Some idea of the pedagogic writings of antiquity, and of the method of Socrates in particular, may be obtained (without reference to the originals or to scattered translations) by means of Saffroy et Noel, *Les Écrivains pédagogiques de l'Antiquité, Extraits des Œuvres de Xenophon, Platon, Aristotle, Quintilien, Plutarque.* Paris, 1897.

The book is very readable, and will not increase the reader's readiness to concede to the "new" views and theories of the day all the novelty which they claim.

to the conclusion desired by a series of easy leading questions. The character of the method can best be shown by an example.[2] (The boy is an illiterate slave.)

The Socratic Method.

Soc. Tell me, boy, do you know that a figure like this is a square?

Boy. I do.

Soc. And do you know that a square figure has these four lines equal?

Boy. Certainly.

Soc. And these lines which I have drawn through the middle of the square are also equal?

Boy. Yes.

Soc. A square may be of any size?

Boy. Certainly.

Soc. And if one side of the square be of two feet and the other side be of two feet, how much will the whole be? Let me explain: If in one direction the space was of two feet and in the other direction of one foot, the whole would be of two feet taken once?

Boy. Yes.

Soc. But since this side is also of two feet, there are twice two feet?

Boy. There are.

Soc. Then the square is of twice two feet?

Boy. Yes.

Soc. And how many are twice two feet? Count and tell me.

Boy. Four, Socrates.

Soc. And might there not be another square twice as large as this, and having, like this, the lines equal?

Boy. Yes.

Soc. And of how many feet will that be?

Boy. Of eight feet.

Soc. And now try and tell me the length of the line which forms the side of that double square: this is two feet — what will that be?

Boy. Clearly, Socrates, that will be double.

Soc. Do you observe, Meno, that I am not teaching the boy anything, but only asking him questions; and now he fancies that he knows how long a line is necessary in order to produce a figure of eight square feet; does he not?

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[2] Plato's Dialogues; Meno. Jowett's translation.

Men. Yes.

Soc. And does he really know?

Men. Certainly not.

Soc. He only guesses that because the square is double, the line is double.

Men. True.

Soc. Observe him while he recalls the steps in regular order. (*To the boy.*) Tell me, boy, do you assert that a double space comes from a double line? Remember that I am not speaking of an oblong but of a square, and of a square twice the size of this one, — that is to say, of eight feet, and I want to know whether you still say that a double square comes from a double line?

Boy. Yes.

Soc. But does not the line become doubled if we add another such line here?

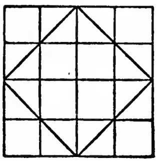
Boy. Certainly.

Soc. And four such lines will make a space containing eight feet?

Boy. Yes.

Soc. Let us describe such a figure; is not that what you would say is the figure of eight feet?

Boy. Yes.



Soc. And are there not these four divisions in the figure, each of which is equal to the figure of four feet?

Boy. True.

Soc. And is not that four times four?

Boy. Certainly.

Soc. And four times is not double?

Boy. No, indeed.

Soc. But how much?

Boy. Four times as much.

. . . .

Soc. Then the line which forms the side of the square of eight feet ought to be more than this line of two feet and less than the other of four feet?

Boy. It ought.

Soc. Try and see if you can tell me how much it will be.

Boy. Three feet.

Soc. How much are three times three feet?

Boy. Nine.

Soc. And how much is the double of four?

Boy. Eight.

Soc. Then the figure of eight is not made out of a line of three?

Boy. No.

Soc. But from what line? Tell me exactly, and if you would rather not reckon, point out the line.

Boy. Indeed, Socrates, I do not know.

Soc. Do you see, Meno, what advances he has made in his power of recollection. He did not know at first, and he does not know now, what is the side of a figure of eight feet; but then he thought that he knew and answered confidently, as if he knew and had no difficulty; but now he has a difficulty, and neither knows nor fancies that he knows.

The Value of the Socratic Method.

As used by Socrates it was generally *destructive*, used to overthrow some false opinion held by the pupil. To-day it is also used by teachers *constructively*, to lead the pupil to formulate a right opinion. Occasionally it may be of good service for this purpose, but its chief value, now as then, is the demolition of the false. A series of skilful leading questions may serve better than any other means to convince the pupil of the falsity of some opinion which he holds; still the Socratic method should be employed with caution. The very leading character of its questions and the consequent passive attitude of the pupil, makes its frequent employment inadvisable.

The Heuristic Method.

Like the Socratic method, the *heuristic method* (named from the Greek word *εὑρίσκω*, *I find*) tells the pupil little directly, but leads him on by questions and problems. It avoids the leading character of the questions of the Socratic method but aims to put the pupil into the attitude of a discoverer, by proposing questions and problems whose replies are not obvious though within the power of the pupil. This method, which is a mode as well, is essentially active and constructive, and deserves a dominating place in mathematical instruction. Owing to its great importance it will receive more detailed treatment in a subsequent chapter.

The system of procedure which has recently been discussed extensively under the name *laboratory method* is very markedly a mode as well as a method. As a method, its characteristics are much emphasis on the inductive genesis of the mathematical deductions; much work with the concrete before passing to the abstract, with the particular before passing to the general. The method will be taken up at length in a subsequent chapter.

The Laboratory Method.

Modes in Mathematics.

A number of modes of instruction may be mentioned: The examination, the recitation, the lecture, the genetic, the heuristic, the individual, the laboratory. The descriptions to be given must necessarily be of the fully pronounced, typical forms: in practice the extreme form of any mode is the exception; they shade into each other, and few teachers use any one mode exclusively.

Modes to be considered.

In the *examination mode* the teacher assigns certain tasks to be done, usually a portion of a text-book to be learned (memorized) or problems to be solved. The class period is taken up by what is tantamount to an examination of the pupils by the teacher, who thus finds out, by means of various tests, whether or not the pupils have performed the task. In its unmitigated form this mode reduces the teacher to little more than a machine. He gives no more help, stimulus, or inspiration to his pupils than the time clock which the workman must punch to record his arrival at his post, or the scales which weigh the result of his toil. In fact, it is easy to imagine a sort of phonograph which would do the work just as well, remaining quiescent as long as the words of a certain text were being said into it, but shouting "Wrong! The next!" as soon as aught else was said into it. It is difficult to think of anything to be said in favor of this method, and it has happily well-nigh gone out of use.

The Examination Mode.

An actual instance of this mode has been described as follows: [8]

[8] Reidt, *Math. Unterricht*, pp. 29, 30.

"A theorem was assigned. Next time the pupils had each to recite its demonstration *verbatim* according to the book. Those who could do this were assigned the next, to be learned from the book. Those who could not had to repeat the first theorem. By and by the pupils had each a different theorem. The class exercise was conducted as follows: On entering the class, the teacher made a signal to the first pupil to say his theorem, then to the second, and so on to the others. By a special signal, the pupils who had said their theorem well were directed to prepare the next for the next time, and by another signal others received the order to repeat their theorem. The teacher prided himself on the fact that he could thus conduct an entire class exercise without saying a single word."

Probably this extreme is no longer to be found anywhere, but it is not certain that all are sufficiently far away from it to warrant taking down the danger signals. Rote teaching has not yet been so thoroughly eliminated as to prevent an educator, an onlooker as far as mathematics is concerned, from saying very recently:

"They both (Latin and algebra) belong to the group of *memoriter* subjects and are reasonably free from any taxing demands upon the higher rational processes." [4]

The Recitation Mode.

The *recitation mode* is a modification of the preceding. As its name implies, it is chiefly characterized by the fact that the pupil "recites" what he has previously learned, and under this mode the class exercise is appropriately called a "recitation."

It agrees with the previous mode in having as chief characteristic that the pupil repeats in the class matter which he has learned by himself elsewhere. It differs from the preceding in that this repetition (recitation) is not regarded by the teacher exclusively as a test, but also as affording oppor-

[4] A. H. Sage, Wisconsin State Normal School, *School Science*, May, 1903, p. 68. An analogous statement was made by Herbert Spencer (*Liberal Education and Where to Find it*). "I doubt if one boy in five hundred ever heard the explanation of a rule of arithmetic, or knows his Euclid otherwise than by rote."

tunity to aid the pupil to a better understanding of the matter in hand.

With the "recitation" there also is often combined some anticipatory work or explanation of the assignment made for the next recitation. The recitation mode may vary from a slight modification of the examination mode, through many possible combinations with other modes. It may at times be used (and with profit) in instruction in which other modes predominate.

The mode of instruction used by any teacher would be classed as the recitation mode whenever it has as its central feature the rehearsal in class of work previously assigned for outside preparation. With this elastic definition, instruction which is valuable, strong, and of high grade can undoubtedly be given by this mode, but it is exposed to serious dangers which must be combated actively and constantly by the teacher. There is a decided tendency to incline too far towards the examination mode, if not actually gravitating into it; the work of both teachers and pupils may become mechanical, and the opportunities for rote and parrot-like work ("recitations") on the part of the pupils are great.

The Lecture Mode.

In the *lecture mode*, the subject matter is presented by the teacher in the form of a connected discourse. The pupils (hearers) take notes which they may afterwards complete and study if they like. The mode is used in the mathematical work of the German and the French universities, and, with modifications, to a large extent in that of the American universities. It is by no means certain that the unmodified lecture mode is the best even for this advanced grade of work, and with rare exceptions it is entirely out of place in secondary work. In Germany, where all the secondary teachers have had at least three years of university training, the danger that they may at times drop into the lecture mode is considerable. In America the danger is growing with the increasing number of men and women of more or less university training who take up secondary work.

In the later years of the course, and in more advanced mathematics, there is larger need for the acquisition of mathematical facts as such, and these facts may sometimes be best learned from books or through direct impartation by the teacher. This would be a passing use of the lecture method, and in such cases an immediate test of some sort should be applied, such as asking the pupil to give back what he has just learned, in order that the teacher may be certain that the matter in hand has been mastered, and to give him an opportunity to correct misconceptions and to strengthen weak points.

The Genetic Mode.

In the *genetic mode*, the subject matter is developed by the class guided by the teacher. All work and think together, the pupils expressing their views as permitted or requested by the teacher, who acts as chairman or leader, assists by questions, hints, and suggestions, sees to it that the discussion reaches the desired result in a reasonable length of time, but allows it all the latitude consistent herewith.[5] All new matter is first developed in the class in this manner.

The teacher is the heart of such work. The text serves for reference, and to obviate the need of taking full notes of the class work. The outside study has as end the fixing in mind of what was brought out in the class, the completion of minor points, the necessary practice and drill in computations.

To test the comprehension of the pupils and to assure their diligence in outside work, the recitation mode may well be combined with the present as one of its minor features. This mode is undoubtedly one of the best, and it is a cause for congratulation that its use is widespread and growing.

The genetic mode implies the heuristic *method*, but there exists a heuristic *mode* as well, which may be combined with the heuristic method. It differs from the genetic mode in the greater stress which it lays upon work by each pupil inde-

[5] Stenographic reports of some class exercises which may be classified under the genetic mode are given by C. S. Osborne in Thought Values in beginning Algebra, *School Review*, 1902, pp. 169–184.

pendently of the others, as distinguished from the class working together as a unit under the genetic mode. It permits a much more important share of the work to be accomplished by the pupil outside the class, and offers greater possibilities of combination with the recitation mode. The latter mode, even the examination mode, may conceivably be applied to matter presented in a text written on the heuristic method. The heuristic mode will be discussed further in a subsequent chapter.

The Heuristic Mode.

The *individual mode* aims to shape the work so that there may be individual progress according to individual strength. In mathematics, progress is conditioned on understanding *everything*. One point left obscure retards progress; several not grasped usually prevent progress. The different rates at which various pupils can work present a very real and serious difficulty to the teacher of mathematics which the individual mode aims to meet. The mode will be discussed in detail in a subsequent chapter.

The Individual Mode.

As mode, the essence of the *laboratory mode* consists in the performing of the bulk, if not all, of the work in the mathematical class-room (laboratory) which should be equipped with appliances for the graphic, the experimental, and the concrete phases of the work. The teacher acts as director of the laboratory, pupils work individually or in small groups, and analogies with the work in the physical laboratory are emphasized. The mode will be discussed in detail later.

The Laboratory Mode.

To characterize some of these modes in a word, it may be said that in the recitation mode the pupil works *before* the class session, in the lecture mode he works *after* it, in the laboratory mode he works *during* the session.

After this enumeration of modes, the question naturally arises, What mode should be used? The good teacher will not confine himself to any one mode. Different modes will be employed at different times, often even in the same class exercise, and procedures will be used which so combine features of various

No one Mode to be used always.

ones of the modes named that they can be classified under none of them. The nature of the topic discussed, the character of the class, the needs of individuals, the material surroundings and class-room equipment, all exercise influence on the determination of the best mode. Not the least potent is the teacher's personality and the stage of mathematical and pedagogic advancement at which he stands. No teacher can select even for himself a permanent mode of handling any subject or topic. The teacher must grow, and next year's view-point may require modification in what is really the most successful mode for him to-day.

The Test of the Best Mode.

Modes are but means; that mode is in any instance the best which in that instance advances the pupil most towards the real ends of his study of mathematics. The teacher must be an active agent in this progress. *If the mode used is such that the pupil makes no more progress than he would have made without the teacher, this on the face of matters condemns that mode under those circumstances.*

This criterion necessitates unhesitating condemnation of the examination mode. Some of the other modes may easily be so handled as to expose them to the same condemnation. The test of any, of every mode, is whether or not it is, at the time when it is employed, the mode which enables the teacher to give to the class, to every pupil, the most of himself, of his knowledge, of his experience, of his guidance, of his enthusiasm, of his inspiration. If at the close of the class hour any pupil can say that the presence of the teacher has been of no help to him in any way, that the teacher has given him nothing, has simply examined him, heard him recite, or allowed him to work by himself, the mode employed with that pupil by that teacher at that time has been a complete failure. The very ease and simplicity of mathematics enables many a pupil to gain a real mastery of the subject even though the teacher gives him no aid, even in spite of teaching which actually hinders, and yet there is no subject in which good teaching so effectively stimulates and aids as in mathematics.

It might not be out of place for the teacher to make an "examination of conscience" at the close of the class hour.

The Teacher's Self-examination.

"Has each pupil profited by my presence in the class-room to-day? Has the mode of instruction employed enabled me to give the class, taken as a whole, more help, more insight, more inspiration than any other mode would have permitted? Is it possible for any pupil to say that he came to my class ready and willing to learn but that his teacher gave him no help? Could the class have obtained all that they got in the class hour to-day equally well from a lifeless book, from one another, or from private study?"

What has been said must not be construed or understood to advocate that the pupil be trained to undue dependence upon the teacher. The preceding chapter will have been written in vain if it has not given unmistakable expression to the thought that the instruction in mathematics should be dominated and determined by the aim to lead the pupil to think for himself. The present chapter simply changes the emphasis: to *lead*, not to *drive*. As soon as the teaching aims at active and independent thinking on the part of the pupil, its constant tendency will also be to diminish dependence on the teacher.

Training to use Books.

The opinion is sometimes expressed that the pupils should learn "how to use books." This is quite true, but so far as mathematics is concerned it would be difficult to think of a practicable mode of teaching the subject which would not train the pupil sufficiently to make all needful use of mathematical books. The subject matter of mathematics is such that the use of a text, or at least of a collection of exercises, is usually advisable; and even in the extreme case that the work is carried on without use of a text so much as for reference, a mathematical proof once understood is recognized in print with sufficient ease to meet all emergencies likely to arise. It is quite desirable that the school library contain various texts, and that the pupils be given specific references to them, and this is quite compatible with any of the good modes of instruction.

CHAPTER IV

THE HEURISTIC METHOD

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Mention should also be made of text in geometry written along heuristic lines, such as those by Dodd and Chace, Hopkins, Sanders, Van Velzer and Shutts, and Williamson.

What is the Heuristic Method?

THE heuristic method (from the Greek εὑρίσκω, *I find*) is dominated by the thought that the general attitude of the pupil is to be that of a discoverer, not that of a passive recipient of knowledge. The pupil is expected in a sense to rediscover the subject, though not without profit from the fact that the race had already discovered it. The pupil is a child tottering across a room, not a Stanley penetrating into the heart of Africa. The teacher stands before him; with word and smile entices him on; selecting the path, choosing every spot where he is to plant his foot, catching him when he stumbles, raising him when he falls, but when he has crossed the room he has done it himself and had made more progress towards walking whither he would than if he had been carried across the room and across hundreds of rooms, or even into the heart of Africa. It is the function of the teacher and of the text so to present the things to be done, so to propose the problems to be solved that they require real discovery on the part of the pupil; that at the same time the steps are within his power, and that he attains in the end a good view of the whole subject.[1] He is led to formulate his own definitions, though the teacher sees

[1] "Children want room to think; their minds have to grow up as well as their bodies. Mental nourishment is quite as necessary as physical nourishment, but it is nonsensical to apply them both in the same fashion . . . anything in the shape of actual teaching or instruction ought to be rigorously avoided. Facts should be regarded as poisons to be used sparingly and with discrimination; every time that a fact is imparted an idea is driven out. There should be no such thing as instruction in the sense which implies the cramming of the brain with information or such mental gymnastics as conjugating irregular verbs and hunting for the least common multiple. The position of teacher and pupil would have to be practically reversed. The pupil would lead, the teacher follow. Boys and girls would then not learn but investigate. The process of learning should be got rid of altogether, being a clumsy, dronish way of acquiring knowledge and one that tends to keep the brain in a perpetual state of dependence.

"Let me give an illustration of what I will call the opposing methods of education. We will suppose, for the sake of the argument, that the only available book for the instruction of boys was that excellent but abstruse work known as 'Bradshaw's Railway Guide.' The modern schoolmaster would draw up an exhaustive and complicated scheme. So much time would be devoted to parsing every sentence through the book; the figures would be added up and subtracted and divided. He would concoct neat little mathematical problems. If the 11.40 express from Paddington travelled to Swindon at 50 miles an hour and broke down half way, to it that they agree with those current; and the teacher also gives him, when he needs them, the arbitrary signs and conventional terms which are in use. The method has been used explicitly in geometry chiefly, but it would do equally good service in arithmetic and algebra.

The Value of the Method.

No discussion is requisite at this point, in addition to what has already been said in the chapter on the purpose and value of the study of mathematics (see especially p. 37) to show that the ends of that study are well attained only if the pupil occupies in the main the heuristic attitude; [2] but no slavish adherence to the method through thick and thin should be cultivated. Thoroughly heuristic teaching may, perhaps should, include didactic presentation of topics which the pupil needs, but which are too difficult, tedious or other-

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at what o'clock would the 12.15 parliamentary train overtake it? and so forth. But — most valuable exercise of all — long tables of trains would be learned off by heart, with the names of stopping places and the prices of first-class tickets.

"A genuine educationist would set to work in a much simpler fashion. He would tell the boys to look out a good train from Birmingham to Newcastle. Each boy would be free to tackle the problem in his own fashion, and the task — if successfully accomplished — would do much towards developing the thinking faculties." — Gorst, Harold E., *Curse of Education*, London, 1901, p. 128.

[2] "[Pupil's] entire nature cries out for action; for the fresh and the vivid. If this inherent craving is vicious, if spontaneity is to be repressed, prescribe a course of lessons in a book opening with abstract definitions followed by deductions devoid of spirit and by pulseless proofs or pseudo proofs, — not that definition, deduction, and demonstration are bad. By no means. These three D's are not only good, they are indispensable, but it is as things not done but to be done, to be produced by the pupil's own spiritual activity, that they are good and indispensable. Lying in the average text they are deposits; evidences and remains of *former* thought. It is, then, this productive life that is to be secured, a life doubly justified, first from within by its own joyousness, and next from without by its results. And what is this life? It is the activity of thought, the play of spirit, dealing not primarily with the D's, but immediately with the stuff to which the D's relate." — Kayser, C. J., *Educ. Rev.*, 1903, p. 248.

wise unsuitable for heuristic development. Care should also be taken to avoid keeping the form while losing the spirit. For example:

False Heuristic.	*True Heuristic.*
Is *A B C D* a parallelogram?	What sort of a figure is *A B C D*?
Is it true that the diagonals of parallelograms bisect each other?	What do you know about the diagonals of such a figure?
Is, therefore, *A E* equal to *E C*?	What lines in the figure are therefore equal?
	How does this knowledge help us in our (main) problem?

The Mode of Instruction.

While the essence of the heuristic method lies in the fact that the pupil is not told things, but led to see them himself, the mode in which this is done may vary. The problems may be formulated for the pupil in a syllabus or in a text-book especially prepared for heuristic work, and the recitation mode of instruction used with it. The individual mode might also be used. An ordinary text may also be used and the matter developed with the class genetically. In the genetic mode, the class *as a whole* discovers the theorems, but no attempt is made so to conduct the work that each pupil independently does so. If, however, the latter is the aim, and the class exercise is devoted to discussing work done and outlining work to be done, we have the heuristic mode in its extreme form.

The method is not without its dangers and disadvantages, some of which should be mentioned here.

Dangers and Disadvantages.

1. Theoretically it would seem that *more time would be requisite* for such rediscovery than for learning and understanding, proofs imparted by text and teacher.

In practice, this proves to be the case at the outset. But little ground is covered during the first month or two, still all who have used the method agree that the power gained in this first slow, but strength-giving, progress makes possible much more rapid work thereafter, and that by the end of the year more ground is covered than under ordinary methods. Before a child can walk, time would be saved by carrying it across the room, but in the end it gets about better by learning to walk.

2. *Heuristic teaching does not mean that the teacher should simply say, "Think for yourself," "Use your brains,"* when the pupil brings a difficulty to him. Such replies are merely admissions of lack of power or will to do his duty. Heuristic teaching is not synonymous with *no* teaching. While as a rule it would contravene the spirit of the method explicitly to tell the pupil the whole solution of the difficulty, he should certainly be *helped*, according to circumstances, by questions, by hints, by outlining a line of attack, by starting the solution. When the difficulties are merely matters of detail or of facts involving no principles, it may sometimes be most helpful to the pupil to tell him the whole directly, especially when his own previous struggles have awakened an eagerness to know, which will itself cause him to appropriate the knowledge so imparted.

It is desirable that the pupil should understand the teacher's aim in not giving a straightforward reply to a civil question, but meeting it with another. Pupils may readily feel that the teacher is very ungracious, unless they appreciate the teacher's motive. In general, the effectiveness of instruction will be enhanced if the pupil knows its ultimate aim and dominating motive, and co-operates in the same spirit, rather than in a spirit of unconcern.

3. *"We cannot expect the average child to be a second Euclid."* Surely this is not to be expected, and has been touched on above. The work must be broken up into simple steps, well within the child's power. And it is easy to be mistaken in thinking the steps are sufficiently simple. The only proper test of this is success of the pupil in doing what is asked, — no matter how clear it is that the child "ought" to be able to do this or that, if he is *not* able (the average child), what more is needed to show that the matter has not been presented to him in sufficiently clear and simple light? Given a child of average intelligence, willingness to work,

desire to please his teacher and to appear creditably before his comrades, it is shifting the blame to the wrong quarter to hold the child responsible if he is not reasonably successful with his work.

4. In the early years (for the heuristic method should begin with the first lesson in arithmetic), the objection may be made that *the child cannot reason at all as yet*, and that it is unwise to try to force him. To force him into the forms of Euclid and of Chrystal, *yes*; to gratify his keen desire to reason, to answer his everlasting "why?", to nourish rather than starve his growing powers, *no*.[8]

5. *Pupils may consult books*, in the belief that it is easier to find and learn proofs in that way than by their own exertions.

This may be discouraged:

a. By frank dealing with the pupils, by telling them the purpose of the method, and by making clear to them that though by the use of books as a substitute for their own thinking the earlier part of the course may be made easier, the rest of it becomes much more than correspondingly difficult; in fact, well-nigh impossible.

b. By not following any one book exactly.

c. By letting the class suggest and formulate problems. So formulated they will not be so explicitly labelled and classified, and the pupil will have more difficulty in deciding where to

[8] "The questions of the child are generally much undervalued as means of culture. The *why* of the child, which parents and teachers often can hardly endure, is completely justified, and should not, as is too often the case, be overheard, left unanswered, or intentionally answered falsely. From the beginning I gave my boy answers to his questions according to my best knowledge, in a form comprehensible to him and in accordance with the truth, and noted that later, in his fifth and sixth, and especially in his seventh year, the questions were always more intelligent because the earlier answers had been retained. If one answers with jokes and tales, or not at all, it is not surprising that even a talented child will ask strange and foolish questions and think illogically, which would hardly occur with correct answers." — Preyer, *Die Seele des Kindes*, Leipzig, 1884, 2d ed., p. 413.

See also Queyrat, *La logique chez l'enfant*, Paris, 1902.

look for the solution, especially the lazy or stupid pupil, who is not interested in the work. It is quite possible that the class may assign itself its own next lesson, especially if the teacher is something of a conjuror, and knows how to make a spectator draw a desired card in the firm conviction that he is drawing just the one he wants.

d. By a great deal of work, selected from many sources, or sometimes manufactured by the teacher, for which no solutions exist within reach of the pupil. This work will be so arranged as to be an organic part of the line of work planned for the class by the teacher, and will offer no difficulty to those who have done the work faithfully, while those who have used undue assistance will find themselves unprepared to grapple with the problems.

There may be much benefit for the pupil from proper reference to the books. After he has worked upon a proof himself, turned it over in his mind until he finds himself at the end of his resources, he may with great advantage seek the solution in some book. Likewise, if he has found a satisfactory solution, it will be of profit and interest to compare it with the solutions of others, as published in various books. It is quite in accordance with the heuristic method to have a good text in the hands of pupils, and others available in the library, and to assign work with the understanding that he is to get the solution from the text-book, if after reasonable effort he has not found it for himself. It is this *effort* which brings the power, successful or not. The main thing to be avoided is that the pupil simply passively imbibes the solution without any active effort to find it himself.

6. *The method requires special preparation on the part of the teacher.* This is undoubtedly true. The teacher must himself have the heuristic feeling, and the method is incompatible with the more mechanical modes of instruction. While heuristic teaching is always the hardest variety,[4] the simplicity of

[4] "Usually, I am sure, the teacher who thinks to let his pupils find out everything for themselves' will find out for himself that

mathematics brings this method within the reach of the teacher with less of special aptitude and training than other subjects require, and mathematics is thus, from point of view of feasibility of teaching as well as of learning, fitted to be the first subject in which the pupil is seriously held to making observations and discoveries for himself.

7. A recent writer,[6] while favoring the method in general, points out:

a. That the danger is as great of telling pupils too little as of telling them too much.

b. That "scientific discoveries have seldom been made in a purely inductive manner. The great investigators were acquainted with the results of others, and usually their work was formulated in hypotheses which they tried to prove by deductive methods. . . . Although the beginner must be placed in the position of a discoverer, it must not be forgotten that the original discoverers had the aid of the observations and opinions of others. It is not more than right that the pupil should be given analogous help, otherwise their observations easily lead them astray."

c. That the instruction of more advanced pupils may be more didactic than that of beginners.

But it must always be remembered that the heuristic method, with all the imperfections which it may have, comes much nearer to realizing the aim and ideals of mathematical instruction than any mere passive ingesting of a body of mathematical

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he has somehow got the hardest part of the undertaking. For visible progress must be made, tangible results must be reached; the teacher must somehow bring things to pass, in spite of the vast capacity of going wrong which marks the efforts of the ordinary individual as it has marked the efforts of the human race to 'find things out.'

"Young people are not averse to games of hunting; but, if the hunting lasts very long without result, most of the participants will fall out, and the game, in school or out, will flag." — Hall, *Physics*, p. 276.

[6] Fischer. See bibliography.

facts, however voluminous. Its failures as well as its successes are on a higher plane than that of mere acquisition of facts and formulæ.[6]

The heuristic method has been frequently tested in the class-room; what is the verdict of experience? Of the considerable number of teachers who have to my personal knowledge used the heuristic method, none have abandoned its spirit. They have modified details, as guided by experience, but only in order more effectively to awaken and cultivate the heuristic spirit in themselves and in their pupils. Numerous teachers testify that the weaker pupils profit specially by the method.[7]

The Verdict of Experience.

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[6] "But the modern English mind has this much in common with that of the Greek, that it intensely desires in all things the utmost completion or perfection compatible with their nature. This is a noble character in the abstract, but becomes ignoble when it causes us to forget the relative dignities of the nature itself, and to prefer the perfectness of the lower nature to the imperfection of the higher; not considering that, as judged by such a rule, all the brute animals would be preferable to man, because more perfect in their functions and kind, and yet are always held inferior to him, so also in the works of man, those which are more perfect in their kind are always inferior to those which are, in their nature, liable to more faults and shortcomings, . . . and therefore, while in all things we see or do we are to desire perfection and strive for it, we are nevertheless not to set the meaner thing in its accomplishment above the nobler thing in its mighty progress; not to esteem smooth minuteness above shattered majesty; not to prefer mean victory to honorable defeat; not to lower the level of our aim, that we may the more surely enjoy the complacency of success." — Ruskin, *Stones of Venice*, II. ch. 6.

[7] "My experience is that the young boy of nine or ten can be readily got to think; the boy who has had considerable training on school lines can only rarely be got to think at all." — Picton, *Great Shibboleth*, *School World*, 1899, p. 397.

In the same connection another writer says: "Undoubtedly the method has its drawbacks. The investigation mentioned above [on the rate of expansion of water when heated] occupied the better part of a term, during which, no doubt, the boys might have read through some little text-book or pottered through a course of ready-made 'experiments' on 'heat.' It also cost the master a great

When the dominating aim is the discovery of proofs, the question of how to find them becomes of capital importance.

How to find Proofs.

There is however, no royal road to the discovery of mathematical results. Original discoveries by the great men of this science no doubt require a special aptitude and training, and involve the element of good fortune as much as discoveries in other sciences. Discoveries in this sense cannot be made by the pupil. His discovery is prepared for him, but enough is left for him to do to give his work the character of real discovery.

General suggestions of what it is well to try will be legitimate help. For example:

It is useful to study the properties of the figure or expression in general, — to determine all there is to know about it. Very interesting class exercises can be made by some such discussion with no definite theorem announced for proof.

Oftentimes, especially in algebraic proofs, it is well to begin with special numerical examples, easy and clear in themselves, gradually generalizing until the theorem desired is reached.

In all cases, constant recurrence to *what is known* and *what is to be found*, a formal enumeration of all data and of the desiderata, is likely to suggest a line of attack. Many a time the difficulty or the clue lies in a connection so simple that it will be considered explicitly only in a careful enumeration of

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deal of labor. But he finds that a very little of this sort of work goes a very long way. It seems to confer a power that is not acquired in any other way. The pupil's mind gains a freedom, a power of seeing things for itself, an alertness and adaptability in turning to fresh matter which make great gaps in methodic knowledge of comparatively little importance. I have more than once been astonished at the ease with which boys who have worked on this plan within a very small range have been able to grasp the bearings of experimental work in quite another department. Their eyes seemed to see things and processes in themselves, and not through the mist of conventional terminology."

All this, though meant for chemistry, applies with even more force to mathematics on account of the greater simplicity of its data and laws.

all the hypotheses. This is one of the values of bringing one's difficulties to others, — a recapitulation of the whole situation is required, and for this reason some good teachers require their pupils to present all their difficulties carefully formulated in writing, the mere formulation often showing the solution, or preparing the way for it.

Neat, orderly work is always helpful, and figures drawn at least fairly accurately are often suggestive.

It is well not to try to correct the consequences of an error discovered, but work afresh. Fresh work will also often disclose an error that is plausible enough not to be recognized in looking over old work.

The question has been repeatedly put to me: "I wish to take up the heuristic method. How shall I do it?" The reply has been:

How to introduce the Heuristic Method.

"Grow into it. Transform your teaching gradually into the heuristic form. Don't begin by throwing away the text-books; heuristic teaching is a very different thing from 'teaching without a text.' The discarding of the text, if it comes at all, should come last, not first. The veteran heuristic teacher may discard the text if he likes, but the beginner certainly should not. Collections of exercises at least should always be in the hands of pupils.

"The first thing to do is to cultivate the heuristic spirit in yourself, and for this there is nothing better than the study of more advanced mathematics, and the intensive and extensive study of the subjects you teach.

"In the class-room use a good text, and follow it as long as you can, while allowing yourself and your class heuristic freedom. It is perhaps well to begin with only the following procedure, which is the corner-stone of the heuristic edifice:

"Take up new matter first in the class, — not didactically, but problematically, as something to be considered and worked out. Aim to make the pupils themselves reach and give you as much as possible of the matter you seek. What you cannot get so, give them yourself. After the class session the book may be used for reference, for review, and completion of what was outlined in the class. The major part of the class period should be given to such work in advance; the first part of the hour may be given to review of the previous day's work, on which in the interim the book has been consulted. Drill problems should also be taken up outside and in the class."

X CHAPTER V

THE INDIVIDUAL MODE

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The Need of Individual Teaching.

It is a platitude to say that teaching should be adapted to the individual, and that the needs of individuals differ. Every good teacher aims to reach his pupils individually, as far as possible; but so long as it remains necessary to allot many pupils to one teacher, the effort will perforce be much hampered and only moderately successful. Attempts have been made to devise modes of instruction which should allow the needs of each pupil to receive fuller individual recognition than they could under the current class system. Some of these plans have been of a general character and applied to mathematics along with the other subjects; others have related to mathematics in particular.

The need of some such system in mathematics is especially great, really almost imperative. If yesterday's work in history, geography, Latin, was not mastered, it will make to-day's work a little harder, but to-day's work in mathematics is impossible until yesterday's has been mastered. In other subjects to-day's success may help retrieve yesterday's failure; in mathematics, when the pupil has lost his grasp at one point he can seldom regain hold at another. The requirement that the pupil master sufficiently all the essential points is inexorable; failure to do so in any important respect bars the way for further advance, and has as inevitable consequence — collapse. In view of the many ways in which a pupil may fall behind a little — through absences, through a topic especially difficult to him, through a personal rate of work slower than the pace set for the class — the task of the teacher of mathematics is most serious. He cannot adopt the plan of teaching for the slowest, without being weighed down by the consciousness that he is holding back all the others. If he adopts an average pace, he has double weight on his conscience; he knows that he has doomed all who might master the subject well if the pace were a little slower, to flounder along practically without hope from the outset, until they finally collapse entirely, and on the other hand, he knows that he is nevertheless holding back all who could proceed at a more rapid pace. Without doubt, a considerable portion of the failures in mathematics can be attributed to causes which could be remedied if the instruction could be adapted more fully to individual needs; especially is this true of that large class of those who manage to "pass," but fail to gain a real insight into the subject, or to enjoy at least a moderate amount of satisfaction and success in the work. If the pupil could have his individual mathematics lesson as he has his individual music lesson, not one out of a thousand would fail.

The Need especially great in Mathematics.

Driven by this pressure, teachers of mathematics have done what they could, under the limitations of the class system, to make individual progress possible for their pupils. Within my own personal acquaintance quite a number of teachers have, independently, thrown off

What has been done.

the class system, and replaced it by a mode of instruction of their own devising, aiming at individual instruction and progress. These teachers have thought out separately, substantially the same mode of conducting the class work, and as very little has been published concerning the mode as used in mathematics, a brief account of its features follows.

The Technique of the Individual Mode.

The essential characteristics of the mode are:

1. Assignment in advance of specific work to be done (theory and problems of a text-book).

2. The pupil works as rapidly or slowly as he pleases.

3. The class exercise is modified as the work progresses. At the outset all are together, soon the class is broken up into groups, then these are subdivided until sooner or later very few if any pupils are together. Explanations and assistance are given to groups when possible, to individuals whenever needed. Some teachers have the pupils assist each other to some extent, during the class hour as well as outside; others conduct the class work much as usual, but no pupil is required to pay attention. The pupil works quietly at something else if he is satisfied that he understands the point under discussion. Other teachers abandon the class exercises entirely and spend the class hour in passing about the room answering the questions of the pupils, each of whom is working on his own problem.

4. The work assigned is divided into convenient sections. When the pupil thinks he has mastered a section, he presents himself for a test. According to the outcome of the test he is assigned supplementary work to do on that section, or promoted to the next.

5. When the pupil has thus completed the work which is customarily done in the subject, he is excused from further attendance on the class, and credited with the subject. The quickest pupils usually finish in about half the time allotted under the class system.

As a result of quite a number of practical tests by different teachers, the principal advantages and disadvantages of the mode may be enumerated thus:

Advantages:

Advantages and Disadvantages of the Mode.

1. There is a fair chance for every pupil, especially the slow pupil who would be crushed under the car of class progress.

2. Each pupil works all the time.

3. Self-reliance is cultivated in the pupil.

4. The pupils do more thorough work.

5. They do more work.

6. Whatever is well done need not be repeated, even though the pupil cannot complete the work. He begins next year or term where he stopped last. Waste is prevented. The work is solid as far as carried.

7. The weaker ones are not carried beyond their depth.

8. The pupils are more thoroughly interested in their work.[1]

9. There is more cordial feeling towards the teacher. He is a friend in need, not a taskmaster, or even a drill-master.

Disadvantages:

1. The benefits of the class exercise are lost.[2]

2. There is a tendency to superficial work by those in haste.

3. Pupils suspend work at times without good cause. Inquiries by several teachers for candid opinions of pupils found

[1] This was very noticeable in the classes of different teachers which I have visited. Every pupil was busy and interested, — undoubtedly more so than they would have been under the ordinary system with the same teachers. They seemed to regard the work as their own.

[2] "The ignorance of intelligent people, even those engaged in the work of education, in this matter (class methods and management) is astonishing. The advantage of the class recitation over the private tutor is not well understood. The class is the most potent of all the instruments in the teacher's hand. He so manages the recitation or class exercises that each pupil learns to see the lesson through the minds of all his fellows, and he learns likewise to criticise the imperfect statements made by them through the more adequate comprehension of the teacher." — Harris, W. T., in preface to Baldwin's *School Management*, N. Y., 1897.

that the weaker pupils each express the opinion that he personally would have done better work under the stimulus of a specific daily requirement.

4. The incessant changes from one part of the subject to another, the unrelaxing alertness required to seize and handle well and quickly the diverse special needs of the pupils, are wearing in the extreme to the teacher.

5. With classes of twenty-five to fifty pupils, the teacher can give to each pupil a maximum of two minutes' time per class exercise.

6. If the teacher wishes to keep in adequate touch with the condition of the pupil, he must require an amount of written work entirely too large for him to correct. The preparation of quite a number of test papers for every pupil (no two papers alike) is in itself a serious task.

The Outcome:

On account of some or all of the disadvantages cited, nearly all of those whom I know to have been using the mode have abandoned it in its extreme form, while in nowise desisting in the quest for the essential end aimed at.[3]

The Essential End.

What is the greatest claim of each pupil as an individual upon his teacher? Is it that he may make as rapid and effective progress as is possible to him individually in traversing the fields of knowledge, or is it that he may come into as close and constant touch as possible with his teacher's personality, scientific and individual, obtaining from his teacher the maximum, not only of knowledge, but of

[3] The individual method is not new. "With a book of his own, the pupil solved the problems contained in it in their proper order, working hard or taking it easy as pleased him, showing the solutions to the master, and, if found correct, he generally copied them into a blank-book for the purpose. Some of these old manuscript ciphering books are still preserved among family records. When a pupil was unable to solve a problem, he had recourse to the master, who solved it for him." — Wickersham, *History of Education in Pennsylvania*.

guidance, of stimulus, of inspiration as well? These two are not mutually exclusive, quite the contrary; but the development of personality by contact with personality, the kindling of the mathematical spirit by the spark from the spirit already aflame, is so decidedly the more important, that no mode can be called good which restricts the opportunities for this individual teaching. The class exercise in which the class is, as it were, one individual and the teacher another, seems still unexcelled as the means of obtaining the maximum of distribution of the teacher's personality to *every* pupil. Every pupil has his attention focussed on the teacher throughout the entire class exercise, and the teacher, directing the thoughts of all, fuses the many into one without losing consciousness of one of the many.

Teach for the slowest if need be, — the slowest properly prepared pupil who is ready and willing to work, — and count your teaching a success if each pupil leaves the class at the end of the hour feeling that he has spent an hour with *you;* but let no number of problems worked or pages plodded through by the pupil make you deem your work anything but a failure if he has received nothing from *you.*

What has the discussion of practical value for the teacher? Much. The problem presented at the outset is serious and oppressive to the earnest teacher. While it is far from certain that the mode outlined meets the difficulty satisfactorily, or that it offers anything that would warrant sacrificing the unity of the class, yet the discussions and experiments deserve careful study for what they suggest for use in connection with class work.

The Pedagogic Result.

CHAPTER VI

THE PERRY MOVEMENT: THE LABORATORY METHOD

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The Movements.

WITHIN recent years there have sprung up in England and America movements which bid fair to exert a far-reaching and beneficial influence on the teaching of mathematics. By rather a curious coincidence it has happened that in each case a marked impetus was given to present activity by an address delivered before the leading mathematical or scientific organization of the country. These addresses, cited at the beginning of the bibliography, are of fundamental importance in a study of these movements. The movements differ in details, but have to the present agreed in their most essential characteristics, and may well be treated together. In America, a term — the *laboratory method* — has been coined (or rehabilitated) to name the teaching of elementary mathematics as it would be if remodelled in accordance with the aims and ideals of the movement, and it is in this sense that the term will be used in what follows. As used here it denotes the totality of the constructive proposals that have been made (and, indeed, that may yet be made) in both movements.

The Keynote — Interest

The Dominating Thought.

The dominating thought of the movements is a fuller consideration of the child mind; a sacrifice of the logical, as hitherto regarded, to the psychological; or, rather, a recognition of the fact that no method of instruction is truly logical which is not psychological, which does not pay heed to the constitution of the child's mind. The assertion is freely made that the centre of gravity of actual mathematical instruction is in the subject matter, in the preparation for examinations, anywhere but where it should be — in the child's needs and capacities.

Underlying all the proposals of the movements, as tacit major premise, is the conviction that a determining test of the matter and the mode of instruction is its fitness to arouse and to hold the child's interest. Before considering the specific ways in which this underlying aim has taken shape the question of interest itself demands fuller attention.

Interest.

That the pupil should be kept interested in his work, and that hence the work should be presented in an interesting way, in the *most* interesting way, may be called a pedagogic axiom, yet thinkers of the most divers sorts accuse the teachers of mathematics of transgressing it grievously.[1] The well-worn figure of appetite and digestion

[1] The problem of interest is not at all peculiar to mathematics, though the accusation may not be without color that the teachers of mathematics have been peculiarly slow in recognizing its fundamental importance. A non-mathematician does not hesitate to voice his observations in this respect when he says:

"There are only two ways to make things stick in the mind of the pupil, one is to repeat the thing without variation until it becomes a habit of mind. This is the method of the algebra teacher. The other is to present the thing with such interest to is still a good one. It has been believed from time immemorial that a good appetite aids digestion, though no longer ago than 1901 the Nobel prize was awarded for researches which by scientific methods ascertained and formulated the relations between the two.[2] The laboratory method aims to arouse teachers to a belief, not only theoretic but practical and effective as well, that mathematical dishes must be made appetizing and palatable if they are to be accepted with pleasure and digested with ease. Of course there are pathological conditions when no food is acceptable, and when nutrition must be taken as a matter of duty and perhaps in positively repugnant form, but these morbid conditions are relatively rare, and the mathematical cook would do well to look to his sauces when his favorite dishes are rejected or swallowed with wry faces.

Mathematics has been called the "whetstone of the wits"; but this is true only of wits that are engaged on mathematics with interest.[3] The sharpening takes place unconsciously.

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the learner that his whole being responds to the act of accepting and adopting it, and with such intensity that with only one or possibly two presentations of the thing it is indelibly fixed in the mind of the pupil. This is the necessary method in all rational subjects." — A. H. Sage, *School Science*, 1903, p. 78.

[2] "Mental fatigue is not only dependent upon the performance of work, but also upon accompanying conditions of the emotions. It is a well-known psychological fact that feelings of displeasure will facilitate fatigue while feelings of pleasure will check it. The inner relation of the individual to the work he is performing is a factor which must be considered in investigating his fatigue limit." — Richter, in "Lehrproben und Lehrgänge," Halle, quoted in *Rept. Com. Educ.*, 1894–95, p. 452.

[3] "It is no use to tell that awkward youth who is always sulkily asking us the wherefore of these triangles, parallelograms, and circles, that they are the whetstones for his wits. He is not aware that his wits need sharpening, nor would he greatly relish the prospect if he were. Indeed he regards his discovery of the uselessness of Euclid as a proof of his already superior sharpness, so we may lawfully use lower motives with him (tell him of practical applications, let him make models and measurements)." — Workman, quoted in Barnett, *Common Sense in Educ. and Teaching*, p. 230.

Mathematics mechanically crammed down the reluctant gorge will no more sharpen the wits than the repugnant beefsteak will nourish the seasick voyager. As a last resort it may be necessary in one case as well as the other to take some nourishing material by mere force of will, but in each case likewise there should be a definite realization of the fact that the condition is abnormal, and that the most fundamental and urgent thing of all is to effect a permanent cure which will restore the appetite.

The Method of Nature.

The teaching of mathematics has not yet sufficiently recognized the fact that nature usually dissimulates her end: that the source of interest and the object to be achieved are usually quite distinct. We eat to tickle the palate, to assuage the pangs of hunger, but the unconscious purpose is nutrition. The child romps to give vent to his impulses, he plays his games to win, for the fun of it, while the unconscious end is exercise and growth.

Even the adult who knows the real objects of his instincts and cravings may usually find the incentive to their gratification in other than a purpose to fulfil their true objects. Though, for example, he may and should consider the real end of eating in planning his diet, yet he need not think of this at the time of eating, but will get much better results by giving himself over to the pleasures of the table, intellectual and physical.

What is the lesson for teaching of mathematics contained in all this? That the curriculum maker, the text-book writer, the teacher, each in executing his functions, should indeed consider the real ends of teaching of mathematics, but that in looking for the clue to interest the child it will probably be futile to seek it, either in mathematics itself, or in the purposes for which it is taught.

This clue must be sought in the child's nature.[4] The child's

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[4] "The difference between the application and zest of boys and girls in the secondary school was well summed up in my presence by a high-school youngster who was twitted by a schoolmate, a girl, on the inferior achievements of the boys as compared with the girls. He said, 'Hm! the girls have nothing else to do.' It

instinct is to *do* — to exert his powers. Because to most adults pulleys are more interesting than permutations, it does not follow that they will be so to the child. Utility often acts powerfully in arousing the interest of the adult; utility alone is never the child's standard. When he appeals to it, asks what is the "use" of this or that, he is sophisticated; he is not expressing his own

The Child's Standard of Interest.

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has seemed to me that the way to enable the boys to see that for the time being they too '*have nothing else to do*' is to connect the school interests with life interests; in other words, so to construct the school's programme that stress is laid throughout on the boys' vocational and social interests, so that these interests shall come naturally and gradually to include the culture interests as well, and this I think is not difficult to do. . . . For example: The future artisan will be interested in the history of his craft; thence easily in the history of industry: . . . History, economics, and government thus become interesting because they may be shown to have an obvious relation to his dominant interest. . . . The obvious dependence of thorough comprehension and pursuit of any trade on mathematics and natural science leads to these sciences. . . . For the future merchant or manufacturer, machinery for manufacture and transportation are incidentally interesting at first because they constitute a part of the vast commercial activity to which the future merchant feels himself irresistibly drawn. Ere long, however, he finds that a comprehension of them depends on a satisfactory knowledge of mathematics and physical sciences. . . . Similarly, the future artist, with his dominant æsthetic interest, may be led to take an interest in science, in mathematics, in history, and in language, because he finds in each of these subjects important assistance towards the cultivation of what he has most at heart." — Hanus, *Educ. Aims and Educ. Values*, N. Y., 1899, p. 98.

Query. What about the boy who has as yet no vocational or other dominant interest? For the normal boy the high school is the scene of development of powers and unfolding of aptitudes. His attitude and that of his parents is expectant and questioning, awaiting what the future will bring forth. Probably the majority of boys enter the high school (perhaps even leave it) without any fixed determination, or even marked inclination, to enter upon any particular occupation. The appeal to vocational interest is at best partial and one-sided.

views, but is appealing to the adult point of view. He never asks "What is the use of playing ball?" and this fact points to what is the child's standard of interest, namely, *doing, successful* doing. "Liking mathematics" is practically synonymous with "ability to do the work as presented." The child does not object to "puzzles," provided he can solve them: he likes them — witness the puzzle department seldom absent from children's papers or columns. "The puzzle instinct" is not to be pooh-poohed, it is fundamental in the matter of the child's interest. The question of usefulness of content, whether as discipline or in practical applications, does not trouble him. The average child regards his teacher, his text-book, as sources from which legitimately emanate things for him to do, and if he can do them, he is pleased and likes the doing. This includes purely intellectual doing as well as manual doing.

The Real Objection to the Abstract.

The sound and sufficient objection to the abstractions of mathematics in early work, to dogmatic definitions, rules, formulæ, strict logical proofs, is simply that the pupil cannot, at that stage and without preparation, *do* (understand and use) what is implied by these things. He is driven to rote work, and his teacher is equally forced to countenance it, perhaps even to require it. Abstractions of mathematics are essentially out of place when, and only when, they cannot be brought within the scope of the pupil's interest.

A Word of Caution.

Those who urge in France and Germany, as well as in England and America, that the abstract mathematics heretofore too liberally administered to children should be replaced by more concrete and interesting work, are urging what is sound and timely; but in heeding the admonition let us all be on our guard lest we fail really to come much nearer to the child's standard of interest by the change. In trying to lead him up to the abstract relations through more concrete phenomena which they underlie, care must be taken that the adult's standard of interest does not once more usurp the place of that of the child. That results are available for the world at large, that they throw light on this or that phenomenon, that the data are taken from this or that business, will do little in itself to interest the child, unless the whole has some direct connection with *his own* experience, his own activity. Disappointment awaits him, for example, who expects that a child who finds some problem dry and meaningless, will suddenly be overwhelmed with interest when he is shown a neat way of doing it with squared paper. Lead the pupil to *want* to work out the problem and the interest is aroused. Fix things so that he *succeeds* (but still has worked hard enough to feel elated by his success), and the interest is held.

What arouses Interest.

It must not be forgotten that the mere assignment of work in the ordinary school routine arouses sufficiently the want to do it in the majority of pupils. If this were not so, much weighing and measuring, drawing and plotting, might fail as signally to interest the pupil as the counting and pondering of more abstract mathematics. Few of the subjects of instruction are *intrinsically* interesting to the child: they must be "*made* interesting," and here is the teacher's main opportunity and obligation. The interest thus aroused is accentuated by the pupil's desire for the commendation of his teacher and the wish to appear creditably before his fellows. The teacher may well make use of these stimuli. In one guise or another they are ineradicable from human nature (adult as well as child). The comprehension of mathematics for its own sake may be a chief end of its study, but it would be as unwise to try to get the pupil to make this his conscious governing motive in its study, to the exclusion of others (for example, of the categories named), as it would for a father who is teaching his child to walk, to deprive it of the desire to "come to papa," and to try to get it to toddle about the room with no other object in view than the only real, physiologic end, — strengthening and training his muscles.

The Correlation of Subjects

The laboratory method is not, however, primarily destructive. It does not merely criticise current teaching as being too abstract, too remote from the child's sphere of interest, too little suited to the workings of his mind, it also presents very definite proposals intended to improve this state of affairs. One of the most important of these is its insistent demand for a closer correlation of subjects, both of the mathematical subjects among themselves and of mathematics with physics.

The Constructive Proposals of the Laboratory Method.

That arithmetic, algebra, geometry, trigonometry should be taught side by side by the same teacher to the same pupils, each helping and illuminating the other, and not tandem, as is the custom in America, has long been urged. That the "water-tight compartments" be abolished, that mathematics be treated as one subject is one of the leading theses of the laboratory method.[5] The topic will be discussed more in detail in another connection. The early introduction, even in elementary geometry,[6] of the elements of analytic geometry and of

Correlation of the Mathematical Subjects among themselves.

[5] Engineers tell us that in the school, algebra is taught in one water-tight compartment, geometry in another, and physics in another, and that the student learns to appreciate (if ever) only very late the absolutely close connection between these different subjects, and then, if he credits the fraternity of teachers with knowing the closeness of this relation, he blames them most heartily for their unaccountable way of teaching them." — Moore, *Presidential Address*.

[6] Tannery, in the *Revue Pedagogique*, July, 1903, advises that the reasoning which is customarily used to the equivalence of the volumes of oblique and right prisms "be kept in an historical museum as evidence of how intelligent our ancestors were."

He suggests two means of replacing the proof. The one (mediocre) consists of cutting the two prisms into thin slices, or making the prisms out of disks of paper. With such models the theorems can be made "clear as day" to the pupils.

"The second procedure, which is excellent, but demands a marked effort, consists in learning some integral calculus before the calculus[7] — such as the plotting of curves, the derivative as measuring rate and slope, the ideas of integration — has also been advocated, and indeed the new French curricula of 1902, effective throughout all France, have extended the minimum of mathematical acquirements so as to include the elements of analytical geometry and of the calculus. (Previously this minimum included only the elements of trigonometry.)

Excellent French texts have already appeared conforming to the details of the new requirements. The works of Tannery and Borel will be found specially interesting and suggestive.[8]

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studying the measurement of these volumes. Integral calculus! In the secondary school!! Yes, I am not joking. The effort needed to learn what a derivative is, an integral, and how by means of these admirable tools surfaces and volumes can be evaluated, is certainly less than the effort heretofore demanded of a child to establish the equivalence of oblique and right prisms, of two pyramids (the staircase figure, you know, that is so tiresome to make), then the insupportable volumes of revolution. Even to-day I don't know the expression for the volume generated by a segment of a circle turning about a diameter. . . .

"To teach what is needed of the differential and integral calculus and of analytic geometry will require, going slowly, perhaps eight or ten lessons. Don't tell me that the pupils will not understand! Why, then do they understand, what they are taught to-day about the volumes just mentioned? After these lessons, a quarter of an hour will suffice to establish the expressions for all the volumes of elementary geometry? And think, besides, of the world of ideas which will open before the pupil, of the multitude of applications which he can make."

[7] "When the elements of arithmetic, of algebra, and of geometry shall have been freed from the multitude of parasitic propositions and reduced to the exposition of directive ideas and essential methods, not only will valuable time have been gained, but also greater clearness of ideas imparted. This will permit the introduction of the elements of analytic geometry and of calculus.

"This whole totality . . . represents what every ordinarily educated man ought to know of mathematics." — Laisant, *La Mathématique*, p. 270 (1898).

[8] Tannery, J., *Notions de Mathématiques*, Paris, 1903; Borel, E., *Algèbre, Second Cycle*, Paris, 1903.

The laboratory method does not stop with demanding the interrelation of the various mathematical subjects, but proposes that the unified mathematics be brought into close relations with physics and with all the applications of mathematics.[9]

Correlation of Mathematics with Physics.

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[9] "As a pure mathematician I hold as a most important suggestion of the English movement, the suggestion of Perry's, just cited, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing, and graphical methods generally, in continuous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by laboratory process, and, on the other hand, by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics." — Moore, *Presidential Address.*

"There seems to be no doubt that very great advantage is gained in the school teaching of mathematics by the practice of introducing the pupil at the earliest stage possible to the point where algebra and geometry join hands, and where mathematics enters into physical sciences . . . the wise teachers accordingly endeavor to connect mathematics in special applications with the natural science studies of his form; never of course venturing on the more abstruse questions, and taking care that pupils do not delude themselves into thinking that they understand what is still dark to them." — Barnett, p. 235.

"The application of mathematics to all the other sciences is the spinal column of our system of studies; it is in fact this application which ensures continuity and progression to the system." — Bertrand, *Les Études dans la Démocratie*, p. 211.

"From our point of view, a pure science is never a complete science. Consequently all rational instruction should contain, side by side with the elements, continued applications, adapted as closely as possible to the theory. The end of these applications is twofold: first, to give the pupil that drill without which a science is never truly assimilated; and then, what is still more important, to give continual opportunity for linking the concrete with the

It has also been held that mathematics should be regarded as strictly auxiliary to the natural sciences, that this purpose should be determining, and that in the consciousness of the pupils mathematics should be only a preparation for physics.[10] The laboratory method goes a step further and makes the most radical proposition so far made anywhere, viz., that mathematics and physics be organized into one coherent whole, the most extreme form of the proposition being that the reorganization be so thorough as to recognize in the secondary school no distinction between mathematics and its principal applications.

It has even been urged that no formal study of mathematics is needed at all, but that pre-collegiate mathematics at least could be developed incidentally in the study of natural phenomena.[11] Though this proposal is extreme, it contains much good; yet the time must come when the child sees that he will save himself much trouble if he makes a mathematical tool and practises with it enough to have a fair amount of skill in its use. The concrete application gives zest to the work, but there must be occasions when the mathematical process itself is a centre of interest.

In this connection "Story of a School" by Johonnot, should

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abstract, to show how one can pass from this to that, which is the final end of science in general." — Laisant, p. 190. See also Woodruff, Teaching of Elementary Algebra from the Standpoint of the Teacher of Physics, *Sch. Rev.*, 1904, p. 127.

[10] For example, Richter, cited in *Fortschritte d. Math.*, 1891, p. 68.

[11] "Opportunities are abundant — at least would be in an ideal school — of teaching by contact with actual phenomena every mathematical idea and principle required to enter upon college studies, — more than the average freshman has, — without the child's knowing that he is studying mathematics, simply by bringing out the full meaning of the phenomena where an opportunity presents itself, and then giving the child, whatever his age, the mathematician's terms and formulæ for that meaning." — Alling-Aber, p. 172.

be read.[12] It is a suggestive account of a step in the right direction, and justly lays much stress on *incidental* teaching.

"Incidental" Teaching.

"'I declare,' said one of our most observant pupils, as he came out from recitation one day, 'the teaching in all the classes is somehow alike. It makes no difference whether we are in natural science, mathematics, or language, we are going the same road, and each lesson throws a new light upon all the others.'

"'Geometry was developed incidentally out of the needs of constructive art and was carried forward slowly as the gradual progress of the pupil called for further application of its principles. It was specially gratifying to witness the cheerful activity of pupils in this line of work so often dreaded and shirked, and to watch the stimulating effect of power gained in mastering a difficult problem.'"

What is Sound in these Proposals?

What is sound in these proposals for correlation? It is unquestionably sound that the mathematical subjects should not be taught tandem; that they should be taught side by side; that physics should not be taught in one year exclusively, mathematics being "dropped" that year, but that it should be taught simultaneously with mathematics throughout the four years of the course, bringing the mathematical theory and the physical application into close juxtaposition; letting the physical experiment often lead to the mathematical problem; that the courses in physics and mathematics should be organized with intention to dovetail; that the ideal would be that they be given by the same teacher, at once a thorough mathematician and a thorough physicist; where such a teacher is not available, two working in harmony and sympathy, each well prepared in his own line, will probably achieve better results than if the double work be undertaken by a single teacher lacking in preparation in one or the other line. All this seems sound, and has already successfully undergone the test of experience. As to obliteration of dis-

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[12] *Pop. Sci. Monthly*, Feb., 1889; also as Appendix to his *Principles and Practice of Teaching*, New York, 1896.

tinctions, distinctions are like fire, useful servants though bad masters. It would be a doubtful gain, even if it were possible, to abolish the distinction between algebra and geometry, mathematics and physics, when breaking down that wall of separation between them which has often been regarded as an insuperable barrier.

Mathematics found wanting.

The proposals just mentioned are backed up by a complaint very common among teachers of physics, that the pupils are not able to apply in physics the mathematics they have previously learned. The statement of Hall[18] may be regarded as typical:

"This leaves the boy free to make all the mathematical errors of which he is capable; and the number and variety of these which he can put into a simple calculation, especially if it involves a trifle of algebra, is the despair of the teacher, — I cannot say the wonder of the teacher, for the phenomenon, remarkable as it is, soon fails to excite surprise. . . . I find upon inquiry that other teachers, not only in this country but in England also, report a similar weakness in their pupils. Mathematical feebleness and fallibility are the birthright of no small part of every class beginning physics. The only question is, what to do about it."

This is not to be wondered at, since no mathematics has been taken up for a year or more. With the attention absorbed by the difficulties of the new work in physics, it would be a marvel if the pupil were to recall the needed formula from the algebra of his first year or the proposition from the geometry of the second. What is to be done? The best solution so far suggested seems to be to develop physics simultaneously with mathematics, to the great advantage of both subjects. The advantage for mathematics is that the physical genesis of many a mathematical problem lends interest and life to it, while the application to physics tends to fix the mathematical results more firmly in mind. This suggestion seems very promising, but it has not yet been to any consider-

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18 *Teaching of Physics*, pp. 287–288.

able extent tested in practice in this country. In France and Germany, however, mathematics and physics have been taught side by side for years, with excellent results, and no change is in prospect.

Quite a number of the papers mentioned in the bibliography contain specific suggestions as to how the work in mathematics and physics can be brought into close and effective connection. Special attention should be paid to the appendix of the Proceedings of the Second Annual Meeting of the Central Association of Science and Mathematics Teachers, 1903,[14] which contains much material for use in the class-room, and to the valuable syllabus prepared by a committee of the Association of Head Masters,[15] describing many quantitative experiments requiring little or no apparatus, which could well be co-ordinated with mathematical work.

Material for Correlation.

A good similar list is found in Armstrong's "Teaching of Scientific Method," pp. 276 *et seq.*

From Concrete to Abstract

Many students of the past and present teaching of mathematics are beginning to believe that the simplicity of mathematics, the fewness of its data and processes, the completeness of its treatment, have led the teachers of mathematics into a grievous pedagogic error, namely, the presentation of the subject from the outset in its finished form. They are beginning to feel that, simple as it is, mathematics should *grow*; that imperfect beginnings are not necessarily *taboo*.

Mathematics should grow.

It is well urged that the teaching of mathematics has too commonly begun with definitions, axioms (first principles), followed by theorems and demonstrations, and last, if at all,

[14] Published separately. Can be obtained from the treasurer of the Association, Mr. E. Marsh Williams, La Grange, Ill., price 25 cents.

[15] Published in British Special Report, II., pp. 414–423.

some "practical applications."[16] It is not meant that the applications were deferred to the close of an entire branch or even of a large chapter, though often this has been done, but that as a rule the complete logical proof of the theorem has preceded the application. This not only ignores the way in which the abstract proofs of mathematics have been evolved, but also overlooks the needs and constitution of the child's mind. The laboratory method proposes that the experimental origin of mathematics be fully recognized; that the pupil be led to feel the need of the mathematical tool through some material experiment he has made or things he has done.[17] Excellent and suggestive lists

Recognize Experimental Origin of Mathematics.

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[16] "It seems almost as ungenerous as it is common among inexperienced critics to attribute the failures of pupils to faults in the teaching, and if I join with others in attributing the apparent failure of mathematical teaching in the schools to the nature of the teaching, it is because on looking through accepted text-books of the fundamental sciences I find no echo of the way in which every science is developed, through patient observation, classification, and induction to the deductive study that is essentially mathematical. I find a careful and logical presentment of the later stages in the development of the subject, interesting to the experienced student looking back over the ground he has traversed, but separated from the beginner by the whole stage that corresponds with the inductive period in the development of a science.

"The teacher and the beginner who is unable to work out for himself the inductive stage move on different planes, and in consequence the student learns his science as he would a foreign language, by the use of his memory and his formulæ. He using the wrong faculties and abhors the study." — W. N. Shaw, in *Glasgow Report*, p. 72.

[17] "Indeed one may conjecture that, had it not been for the brilliant success of Euclid in his effort to organize into a formally deductive system the geometric treasures of his times, the advent of the reign of sciences in the modern sense might not have been so long deferred."

"The mathematician with the catholic attitude of an adherent of science in general (and at any rate with respect to the problems of the pedagogy of elementary mathematics there would seem to be no other rational attitude) will see that the boy will be learning of such experiments have been published (see the bibliography and p. 103). Experiment, estimation, approximation, observation, induction, intuition, common sense are to have honored places in every mathematical class-room in which the laboratory method has sway.[18]

No doubt, the child's intuition may occasionally play him false and lead him to accept as true what is not true, but this is really merely the teacher's opportunity to help him to make an important step forward and to introduce him in an informal and interesting manner to some strict mathematical reasoning. The danger that the essential mathematical kernel will not come to its rights is probably more apparent than real, provided the teacher steers consciously and intelligently for the proper goal.

After consideration of a sufficient number of special cases it

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to make practical use in his scientific investigations — to be sure, in a naive and elementary way — of the finest mathematical tools which the centuries have forged; that under skilful guidance he will learn to be interested not merely in the achievements of the tools, but in the theory of the tools themselves, and that thus he will ultimately have a feeling towards his mathematical work extremely different from that which is now met with only too frequently, — a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions." — Moore, *Presidential Address*.

[18] "Mathematics ought not to be taught exclusively on the antique model, as a pure science, but, on the contrary, in the modern spirit, as a science at once pure and applied. We have seen with what care mathematics must be made to spring from the crude experiences of the early years: we shall now show that no less systematic and detailed pains must be taken to connect mathematics with scientific experience. I pass no opinion on the origin of the mathematical notions, whether they arise from experience or are innate: this is a problem for metaphysicians. But I know that it is extremely regrettable to aggravate further the abstract character of mathematics by separating it from ordinary experience and from scientific experience." — Bertrand, *Les Études dans la Démocratie*, Paris, 1900, p. 206.

is a relief, a simplification to abstract, to generalize. Abstractions and generalizations are rather the crowning products than the foundation stones.[19] From the point of view of the laboratory method the pupil, when weighed down by the burden of many similar concrete or numerical cases, may be easily led to see that they can all be replaced by a single, though necessarily more vague, case. He thus abstracts his own mathematics.[20] The power of abstraction so developed, very simply and gradually at first, must be used more and more freely. The laboratory method, while insisting on the experimental,[21] the concrete, the workshop

Pupil abstracts his own Mathematics.

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[19] "General formulas which men have devised to express groups of details, and which have severally simplified their conceptions by writing many facts into one fact, they have supposed must simplify the conceptions of a child also. They have forgotten that a generalization is simple only in comparison with the whole mass of particular truths it comprehends, that it is more complex than any of these truths taken singly, . . . and that to a mind not possessing these simple truths it is a mystery. Thus confounding two kinds of simplification, teachers have constantly erred by setting out with 'first principles.'" — Spencer, *Educ.*, chap. 2.

[20] "The essential end towards which human science tends is the study of the phenomena which the external world presents to us. In this study there are three distinct steps:

"1. By abstraction, the magnitudes must be prepared for mathematical study: for a phenomenon too complexed to be fathomed is substituted one more simple in that it relates only to abstractions while still representing approximately (but only approximately) the nature of the facts. This may be called the *putting into equations.*

"2. *The solution of the equations.* This is the purely mathematical step.

"3. *The return from the abstract to the concrete.* This includes expression of the results in concrete terms, discussion of the results and their experimental corroboration." — Laisant, pp. 20–21 (condensed). Note how the abstract is embedded in the concrete.

[21] "The arithmetical problems arising out of these experiments, founded as they are on their own data, are worked by the pupils with an amount of interest, not to say eagerness, which artificially made problems can never inspire." — Smith, *Teaching of Chemistry*, New York, 1902, p. 121.

side [22] of mathematics, and while at present laying *special* emphasis on these things because they have been too long under-emphasized or entirely ignored, is no foe to abstract mathematics, and does not aim at eliminating any abstract mathematics simply as such. Its ideals in this regard will be fully reached when the pupil's attitude is active, not passive; when he abstracts mathematics, but does not simply gulp down previously abstracted mathematics; when the order of instruction is from the concrete to the abstract, and not from the abstract to the concrete.

What is the Concrete?

It may be well to define a little more closely what is that concrete which is thus demanded as the basis of work in mathematics. Is it material things, experiments with them, reasoning about them? Yes, but not these exclusively. It is, as used here, whatever is thoroughly mastered intellectual property of the pupil. At the very outset integral numbers have no sense until they have been exemplified by concrete objects. As soon as mastered they have become concrete and can be used to lead up to new abstractions. In studying properties common to all polygons, — properties of the *n*-gon, — hexagons and octagons may well give the concrete beginnings. The concrete at any stage includes all the abstractions previously made and assimilated. While certain material experiments, the study of certain phases of physical sciences, lead up to a large amount of mathematics in a natural and interesting way, in other cases concrete beginnings may be made as effectively from objects of thought,

[22] Armstrong, *Teaching Sci. Method*, London, 1903, cites statements to show that mathematics is overestimated in British military schools, and then proceeds to set up as "Reasonable standards," p. 64: "It should be insisted that fundamentals be thoroughly taught by practical methods, so that the knowledge acquired may be real and usable: it is astonishing how far students may be carried in mathematics, how real and interesting the subject becomes to them, when they grasp the fact that it has a practical bearing." All of which the teacher of mathematics may well lay to heart.

concepts, previous abstractions, — without any manual activity whatever.[23]

Mathematical Drill needed.

When the mathematical problem has been evolved from the concrete cases, it is still but a problem. There is interest in solving it, for the pupil knows a reason for its existence. The mathematical processes involved must now become the focus of attention and must be mastered with little if any concrete assistance. No one has yet succeeded in devising a method whereby one actually learns the requisite mathematics in the performance of material experiments. Such experiments in so far as they involve mathematics, either serve to apply mathematics already known or to formulate new mathematical problems for study. In either case, the actual mastery of the mathematical process is not a by-product of the material experiment, but requires separate study. In the mathematical work, cases which are concrete in the sense of having already been mastered and assimilated, will be of marked help. Thus pupils might be led to the general solution of quadratic equations by being asked to solve in turn:

$x^2=4$	$x^2+10x+25+2=7$
$x^2=7$	$x^2+10x+34=12$
$(x+1)^2=9$	$x^2+4x=1$
$(x+3)^2=10$	$x^2+3x=7/4$
$x^2+2x+1=16$	$x^2+2ax+a^2=b$
$x^2+8x+16=2$	$x^2+2ax=b$

Such procedure is of the very best for reaching general mathematical results by a march in essence concrete. The mathematically concrete may lead by gradual process and by a fairly direct path to the mathematically abstract. (The physically

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[23] For example: To introduce a college class to the notion of determinants, it would be beginning in a sufficiently concrete way to solve simultaneously the equations:

$$ax+by=e$$
$$cx+dy=f$$

and study the form of the solution.

concrete rarely if ever leads directly to the mathematically abstract.)

Some other Changes urged

So far, the discussion has been restricted to the two important general reforms, — concrete beginnings, correlation of subjects, — at which both the English and the American movement aim. It has been necessary to be summary; for details the reader should refer to such of the books and papers named in the bibliography as may be accessible. A brief mention of some of the other important changes urged must suffice here.

I. *Work with a large body of axioms.* It is urged that all those things should be taken for granted which seem evident to the pupil or whose truth can be satisfactorily established by direct observation, and that the philosophic questions should be deferred or omitted. Let the standard for rigor of proof be the pupil's capacity to appreciate rigor, and not the strictest rigor thus far attained. It is an error to make the pupil learn forms of demonstration for which he sees neither need nor use. All this is sound, and a pedagogic consequence of the strict investigation of axioms of recent years. The whole subject will be discussed more in detail in a subsequent chapter. One word of caution may be spoken here. It is not meant that plausibilities be palmed off on the pupils as proofs. The meaning is:

i. Accept without demonstration things which are true and which the pupil believes without formal proof; for example, that all right angles are equal, that from a point in a plane, one and but one perpendicular can be drawn to any straight line in that plane.

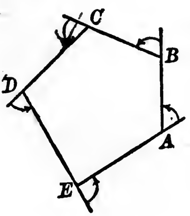
ii. Accept without demonstration theorems which there is occasion to use, but whose rigorous proof is too difficult for the pupil, or which would consume too much time. It is not necessary that every theorem used should have been previously demonstrated. "A child need not know how to make a watch before he may use one" (Perry). The theorem may either be stated on the teacher's authority alone, or with such aid to make it seem plausible as the teacher may think of value, always taking care that the pupil understands that he has not been given the rigorous proof. Absolute truth toward the pupil is the paramount essential.

II. *Various modes of proof to be admitted.* In addition to acceptance of a large body of facts without proof, proofs of types different from those usually accepted may be admitted:

1. *Intuitional proof.* Intuition is sometimes a very satisfactory mode of proof, and the power of intuition should be cultivated. For example: the perpendicular is the shortest distance from a point to a straight line. (The customary "proof" is merely the same intuition concerning double the distance.)

2. *Proof based on measurement.* The measurements may be of material objects or of carefully constructed figures. The result is only approximate, but there is usually a certain part of it which one *knows* to be correct as far as it goes, in consequence of the degree of accuracy of the constructions and measurements. Thus, one can easily prove conclusively by measurements that $3 < \pi < 4$.

3. *Free use of motion,* and of any other aids which help to convince the mind of the correctness of what is asserted. The cast-iron form of the traditional proof is not abandoned, but simply not used exclusively. The following may serve as a specimen of what would be accepted as sufficient proof even from a conservative point of view:



To show that the sum of the external angles of a convex polygon is 360°, suppose a man to walk around the polygon. At each corner he turns through the external angle at that corner, and when he reaches the starting point he has made one complete turn.

Similarly, translation of figures on squared paper is freely used; in fact, all the needful properties of squared paper are unhesitatingly taken for granted.

III. *Teach through the eye.* This may be done by careful drawing, by use of colors, by use of square paper, by careful record of work in a laboratory record book, by schematic outlines of methods and results. The best pieces of apparatus in the mathematical laboratory are pencils and paper: various colors, squared paper.[24] With this simple apparatus much mathematics can be made concrete, ideas of variation, of approximations to limits, of integration even, can be nicely developed, as well as other theorems directly proved.

Graphic Methods.

The graphic representation of numerical facts and relationships by means of curves is important in mathematics and is interesting to the child long before he reaches the secondary school. It is familiar to him from the newspapers and magazines of the day, in which he may see pictures of men of different sizes used to represent the relative strength of the armies of different nations, cubes of various sizes to represent their output of iron, or trains of various lengths to represent their relative railroad traffic.

The principles underlying this representation are understood without any trouble. All the needful properties of the cross-section paper are sufficiently obvious and may be assumed without proof. The use of curves to represent variations of temperature, of price, of population, etc., is already also familiar to him, and its principles are tacitly assumed without difficulty.

To draw such curves interests the child, not only because it gives him something which he can do, but also because the curves themselves are interesting, as showing much more clearly than do tables of numbers how the quantity represented varies, when it increases, when it decreases, at what rate, etc. Two or more curves in the same drawing permit comparisons between the variations of different quantities.

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[24] "Squared paper, a marvellous instruction which ought to be in the hands of every one who works in mathematics from the kindergarten to the university." — Laisant, *L'Éducation fondée sur la Science*, Paris, 1904, p. 23.

Such curves can be drawn with profit as a part of the work in arithmetic, and other graphic representations of numerical facts may also be used, such as by lengths of lines, by squares, by rectangles, by circles. All this work has also marked value as drill in the varied computations required, and in the use made of properties of geometric forms. The data employed would be specific numbers, either given by the teacher or ascertained from actual conditions by the pupil.

In the secondary school, laws may be given from which data themselves may be computed. For example:

Make a graphic representation of railway fares for journeys of various lengths, the rate of fare being 3 cents per mile. From the diagram read off the fare for 7 miles; the distance one can travel for 45 cents, etc.

Many such questions can be asked, both about curves traced according to a given law, and about those traced from empirical data.

The converse process is equally important: given the curve, to determine the law, or read off the data. This can be done occasionally, using straight lines and circles for determination of the law and curves drawn at random for reading off the data.

After a large number of simple problems like those mentioned above have been solved, it is an interesting exercise to ask each pupil to make a problem of a certain type and also the graph representing the relationship. When the graphs and the problems have been handed in, the graphs only are redistributed among the class, each pupil endeavoring to determine from the graph received the mathematical relationship which it represents. Such work can be done most satisfactorily after some study of geometry.

Geometric Construction of Graph.

The construction of the curve can often be accomplished decidedly more expeditiously by the geometric properties of the relationship, than by plotting numerical values.

As example consider:[25]

$$y = x^2,$$

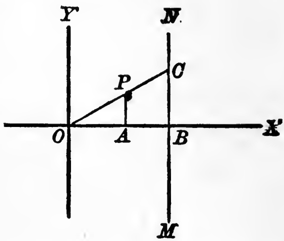
Writing this $\frac{y}{x} = \frac{x}{1}$,

suggests the following geometric construction:

Since $\frac{PA}{OA} = \frac{CB}{OB}$,

if we let $OB = 1$, and $CB = OA$, we have, using the ordinary notation of x and y axes,

$$\frac{y}{x} = \frac{x}{1}.$$



From this an easy practical construction for the graph of $y = x^2$ follows.

Let MN be parallel to the y axis and at the distance 1 from it. For any x lay off $BC = x$. Draw OC. At A, the extremity of x, draw a perpendicular to the x axis. The point P common to this perpendicular and OC is a point of the locus.

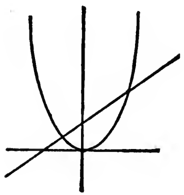
On squared paper, points of the locus can be marked very rapidly. Use a ruler as OC. Put a pin at the point O, and keep the ruler constantly touching it. Let the point C be in turn the successive divisions of the squared paper. The corresponding point A will be in turn the successive divisions along OX. In each position run the eye from the point A along the line of the squared paper to the ruler; mark the point where the line meets the ruler.

Many points of the curve can thus be rapidly marked, and the curve sketched through them represents quite closely the relation $y = x^2$.

[25] From Moore, Cross-section Paper as a Mathematical Instrument, *School Review*, 1906, p. 317. This paper gives a number of other examples, and will repay careful study.

The curve thus drawn is a graphic table of squares; also of square roots, for $x = \sqrt{y}$.

It can also be used to solve quadratic equations. For, put the quadratic equation into the form:



$$x^2 = p\,x + q.$$

Lay the ruler in the position

$$y = p\,x + q.$$

For any point of the curve

$$y = x^2.$$

For the common points of the curve and the ruler,

$$x^2 = p\,x + q.$$

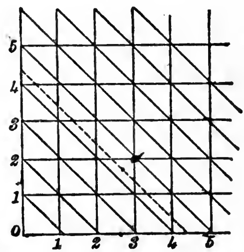
That is, the x's of the points of intersection of the ruler's edge and the curve are the roots of the quadratic equation.

Addition, subtraction, multiplication, and division tables and others may also be represented graphically.

The figure will suffice to show how a graphic addition table may be made.[26]

Graphic Computation.

To add 3 and 2; run to the right 3, up 2; the diagonal line will lead to the sum. The sum of 2.7 and 1.6 may be found similarly by running 2.7 to the right then 1.6 up. The end point does not fall on a diagonal line but the eye will readily imagine one; it may be assisted by a ruler.



Varied Uses of Squared Paper.

These examples will suffice to illustrate the varied and fundamental ways in which squared paper may be used in elementary mathematics. It well deserves the "canonization" which has been proposed for it, and the widespread interest that is being taken in the subject

[26] From Moore, *l. c.*, where others are given.

of graphs is promising. It is encouraging to find elementary text-books making use of graphs, but the possibilities of squared paper are far from exhausted if it is used merely for the graphic solution of two simultaneous equations, linear or quadratic. Geometric construction of algebraic relations, graphic computation, exhibition of tables are also among its possibilities. It serves, as perhaps nothing else can, to weld together the three characteristic mathematical ideas of number, form, and formula under the fundamental concept, *functionality*.[27]

The use of squared paper for the graphic solution of equations is neither the easiest nor the most important of its uses. It is perhaps the most abstract and a large amount of work with concrete data should precede any attempt at the representation and graphic solution of abstract equations. It is a long step from problems like the first of the following to the second:

1. The daily profits of a factory are $2 per workman, less a fixed operating charge of $10 per day. A second factory makes $3 per workman, but its fixed operating charge is $25 per day. Represent the net profits of each factory according to the number of workmen. If the factories employ the same number of workmen, for what numbers of workmen will the first factory make the larger profit? The second? For what number of workmen will they make the same profit? What is this profit?

2. Solve graphically:

$$y = 2x - 10$$
$$y = 3x - 25.$$

Pupils may solve the first with ease and still find serious difficulty with the second. It requires very slow development and considerable training to get the pupils to grasp clearly the idea of an equation in two *variables* as recording a restriction to which these variables are always subject in their variation, and the curve as the representation of their variation under this restriction.

[27] See Moore, *l. c.*

This idea is indispensable in later mathematics. Each teacher must judge for himself whether, and to what extent, it is likely to prove helpful in his own teaching of secondary mathematics. Let no one teach graphic work merely for the sake of "bringing in graphs." Its use is called for when, but only when, the teacher feels sure that such work will illuminate the subjects taught; when the teacher is convinced that he will teach more effectively by its use than without.

When to use Graphs.

IV. *No daily allotment of work.* A minor feature of the method, though a conspicuous one when used, is that there is no specific daily allotment of work, the same for all each day, but a general allotment on which each pupil works according to his strength. This phase of the method is analogous to the individual mode already discussed. A difference between the two is that the laboratory method requires the bulk of the work to be done in the class-room. To attain the best results this necessitates that the class period allotted to mathematics be at least doubled, with corresponding diminution of the outside work required.

V. *Relations between teacher and pupils.* The relations between teacher and pupils are to be made of the most informal character: the teacher is to be leader and friend; the pupils to assist each other on occasion, working singly or in groups as may be best.[28]

The laboratory method in mathematics implies the existence of a mathematical laboratory. Every class-room in mathematics may be regarded as such, but a room specially fitted up with drawing instruments, suitable tables and desks, good blackboards, and the apparatus necessary to perform the experiments of the course

The Laboratory and its Equipment.

[28] "Instructors may fear that the brighter students will suffer if encouraged to spend time in co-operation with those not so bright. But experience shows that just as every teacher learns by teaching, so even the brightest students will find themselves much the gainers for this co-operation with their colleagues." — Moore, *Presidential Address.*

is really essential to the best success with the method. Plenty of working space is desirable. More in detail, the following equipment may be mentioned, arranged roughly in order of importance.[29]

1. A good library, containing a selection of the works named in Chapter IX. In addition it should include quite a number of current text-books; also various tables of logarithms, ranging from 3 to 7 places; interest tables, tables of factors, squares, square roots, reciprocals.

2. Sufficient blackboard to accommodate all the class at once; some cross-section board; spherical blackboards, large and small; hinged blackboards (dihedral angle); blackboards forming a right trihedral angle; others forming the eight octants about a point.

3. Mathematical models, purchased or home-made, or both.

4. Logarithmic slide rules.

5. Surveying instruments.[30]

6. Balances, steelyards.

7. Pendulums.

8. Levers, pulleys, wedges, screws, vises.

9. Mercurial barometer, thermometer.

10. Instruments for measuring specific gravity of liquids; for example, a lactometer.

The term "Laboratory."

Various features suggest the use of the term "laboratory" to designate the method.

1. The fundamental use of material activity, including drawing. Generalizations from the concrete to the abstract.

2. The performance of the bulk of the work in the classroom (laboratory).

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29 Lists are also given by Myers, *School Review*, 1903, pp. 737–739.

30 Excellent directions for the cheap preparation of home-made instruments are given in Myers, *Observational and Experimental Astronomy*, Chicago, 1902.

3. The assignment of a list of things (analogous to lists of laboratory experiments) to be done by all pupils, though not necessarily simultaneously.[31]

The phases which give most direct occasion to the designation "laboratory" are suggested by the physical laboratory, and in this connection there is also a possibility of danger. While all good suggestions from the methods of the physical laboratory are of course to be adopted, caution should be exercised not to adopt its drawbacks as well, or to take for granted that everything done there ought to find its strict counterpart in mathematics.

Limitations of the Physical Laboratory.

In particular, the fact that in the physical laboratory students work singly or in small groups under the general supervision of the instructor, but with direct contact with him for only a few minutes, is a *limitation* of the physical laboratory, not an advantage. It is a consequence of the character of the apparatus which must be used, and it seems that the great advantages of laboratory work in physics can be attained only under this limitation, but it is a limitation nevertheless.

The character of the apparatus — its variety, costliness, and bulk — forbids much duplication, and effectively prevents simultaneous work by all the pupils on the same problem. Class work is not found in the physical laboratory, not because it is intrinsically desirable to exclude it, but because it is impossible to attain it. The laboratory system of physics is far from satisfactory in itself; it simply makes the best of the situation. Every effort is made by physicists to utilize to the utmost the

[31] "A mathematical laboratory is something not often mentioned, but I think the thing, however we name it, a necessity. It will contain, in part, such things as relate to ordinary, not purely scientific, measures." — Safford, *Mathematical Teaching* (published 1886) p. 43.

In the same work (p. 13) J. W. Dickinson, Secretary of the Massachusetts Board of Education, is quoted as recommending "laboratory work in mathematics as well as in physics." On "laboratory" methods in the teaching of history see Bourne, *The Teaching of History and Civics*, New York, 1902, pp. 172–173.

teacher's time in the laboratory. Even so, laboratory work must go hand in hand with class-room work, whose mode of class-room treatment is substantially the same as that of mathematics.

The Situation better in Mathematics.

In mathematics the situation is much better. The possibility of doing much work with the class as a unit is one of the great advantages of the mathematical subject matter, and is to be cherished with the deepest solicitude and not to be abandoned because of a mere analogy. The apparatus needed for mathematics (pencil and paper the most prominent) can be improvised by the pupil as a rule, and it is quite possible to arrange matters so as to retain the benefits of the class system, and add thereto the good features of the physical laboratory system, while avoiding its defects.[32]

The fundamental need here, as ever, is that the teacher *teach*. That he run neither to the one extreme of simply "hearing recitations," nor to the other, of setting tasks to be worked by the unaided pupil in the "laboratory." In fact, the two tendencies, while apparently running in opposite directions, come near to meeting on the other side of the circle.

A Mode proposed.

The procedure which physicists find best pedagogically suggests a plan for mathematics; namely, not that the mathematical class exercise be *supplanted* by a mathematical laboratory exercise, but that it be *supplemented* thereby. Let the mathematical class exercise be conducted in some good mode as at present, with the usual time allotment. Let this be supplemented by work in a well-equipped mathematical laboratory, either under the direction of the teacher or one or more competent assistants, or both. In

[32] "It is important that the subject itself should not be obscured by the multiplicity of its applications. It is true that for the learner these applications are invaluable; but it is a pity when a really capable boy spends all his time in solving 'catchy' problems, when he might be going on with the study of things vastly higher in importance." — Mathews, in Spencer, *Aims and Practice of Teaching*, Cambridge, 1897, 191.

the laboratory work there should be the freest interchange of thought and assistance, the laboratory period to take the place of outside study. The pupils should do substantially the same work in the laboratory, and the class exercise should prepare directly for it.

This proposal recognizes the laboratory work for what it really is, — private study under guidance, with the best facilities and in the most helpful environment. It is directly in line with an important recommendation of the Committee on College Entrance Requirements; namely:

"XIV. *Resolved*, That we recommend an increase in the school day in secondary schools to permit a larger amount of study in school under supervision." [83]

It may be well to recall that when the subjects are duly correlated, physics is in an important sense a laboratory side of mathematics.

The feeling that mathematics must be made more concrete, must come into closer touch with the realities about the pupil,[84] is growing in both Germany and France as well as in England and America, and the influence of the work of Perry

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[83] Report, p. 40.

[84] "No teaching is so difficult as that of mathematics, since the large majority of the pupils is decidedly indisposed to allow itself to be harnessed in the rigid framework of logical conclusions. The interest of young people is much more easily won if one sets out from material things and gradually leads on to abstract formulations. . . . All this is certainly most true, but there lies a danger in it. . . . It is possible that through the mere mass of interesting applications the real logical training may be crippled, and under no circumstances may this happen, for then the real marrow of the whole is lost. Hence, we desire emphatically an enlivening of instruction in mathematics by means of its applications, but we desire also that the pendulum which in earlier decades perhaps swung too far in the abstract direction should not now swing to the other extreme, but we wish to remain in the just mean. To preserve the just mean is the problem and the art of the teacher, which should be furthered by an improved preparation of teachers." — Klein, *Ueber den mathematischen Unterricht an den höheren Schulen*, *Jahresber. d. deutsch. Math. Ver.*, 1902, p. 669.

can be distinctly marked in the current of thought on the European continent.[85]

Similar Tendencies in Germany and France.

Klein tells us, and adds his own weighty approval to the thought, that Schellbach, who did perhaps more for the pedagogy of mathematics in Germany than any other man, "never insisted in a one-sided way on the abstract side of the science, but always emphasized its connections with the problems of astronomy and physics, as well as the general requirements of life: he always regarded the power to formulate mathematically problems coming from without, and the estimation of the magnitude of the various influences at work, as just as important as the strict treatment of ready-made mathematical problems."[86]

The resolution of the "Verein zur Förderung des Mathematischnaturwissenschaftlichen Unterrichts," 1891, is also worth citing:

"The pupils of the secondary schools are in general not sufficiently able to recognize the mathematical phases of the experiences of life, and the reason lies especially in the fact that the applications of mathematics are often made in artificial examples instead of bearing on the circumstances of real life. Hence the system of school mathematics must be built up in its details with view to its natural applications (physics, chemistry, astronomy, business arithmetic, etc.). The examples to be used are to accustom the pupils to observe in the objects of sense perception not only the qualitative but also the quantitative side, to the extent that such a mode of regarding their environment shall become a permanent instinctive need."

And finally, the Royal Order of November 26, 1900, which extends to scientific secondary schools the recognition and privileges theretofore accorded only to the classical institutions, insists specifically on more emphasis on intuition, experiment, and applications in the natural sciences.

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[85] For details as to recent activities see the references of note 29, p. 48.

[86] Klein, in *Lexis, die reform der Höh. Sch. in Preussen,* p. 258.

CHAPTER VII

MISCELLANEOUS POINTS OF METHOD AND MODE

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In the present chapter a number of somewhat isolated points will be taken up briefly, which could not well be discussed in other connections.

Character of Chapter.

Rigor. With regard to rigor there are two extremes,— absolute scientific rigor on the one hand is attainable only by the specialist, and any attempt to approximate to it serves merely to make the subject dry, repellent, incomprehensible to pupils. On the other hand, the crowning glory of mathematics is its peculiar clearness, the strictness of its proofs, and unless the subjective effect of this is preserved some of the ends of instruction in mathematics are not attained.

On the Degree of Rigor to be observed in Class Work.

The best course to be taken seems to be to base the work on *large* bodies of hypotheses, assuming many things which might be proved, but to make inferences from these hypotheses in strict accord with the laws of thought (logic).[1] This does not mean that everything should be built up by strict deductive reasoning on the basis of the premises assumed, nor that the really deductive reasoning used should always be expressed in the most formalistic manner, but that when professing to be strict, the reasoning should really *be* so, and that there should always be a clear line drawn between what is proved and what is assumed as sufficiently plausible without proof, or accepted and used on the assertion of others (*e. g.*, teacher) that it can be proved. Absolute truth towards the pupil is the first essential.

Let the idea of building up a system of propositions rigorously deduced from a set of irreducible assumptions (axioms) be abandoned, but let the distinction between what constitutes a *proof* in mathematics and what does not be all the more

[1] See Chapter X., On axioms.

emphasized. Exactly this frank admission that particular propositions are accepted on authority, on verification, or on other grounds which make them plausible, opens the door for such emphasis. Nothing could be more pernicious than the palming off of sham "proofs" on pupils as real,[2] but with frank admission of unproved propositions the work can be carried on in a stricter and more correct spirit from the outset. When this is done there will be many and large gaps

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[2] "I am convinced that less harm is done nowadays by teaching by rote than those ill-advised attempts at rational instruction which are inspired by imperfect knowledge. Thus, for instance, if you state the binomial theorem for a fractional exponent and the conditions for its validity, you do not educate your pupil, but you give him a piece of information which he can learn to apply and which may be practically useful to him; but if you go on to make him learn one of those unsatisfactory 'proofs' of the theorem which still keep their place in some of the text-books, you are doing positive mischief, and replacing harmless ignorance by a new pretence of knowledge." — Mathews quoted (no ref.), Barnett, pp. 229.

"There is a great deal of talk nowadays about the educative value of the sciences, and it is understood that the value of mathematics lies in teaching how to reason. But surely this virtue is not found in the mechanism of calculation. . . . There is danger in attempting to persuade pupils that they understand operations on the basis of insufficient theory. If the teacher is sufficiently dextrous to accomplish this, he warps the minds of his pupils. A mark of a good thinker is that he distinguishes clearly between what he comprehends and what he does not comprehend; to persuade an individual that he sees clearly when he sees but vaguely is not good; far better that the teacher ask the pupils to accept his word. Those pupils who have a strong and indocile mind, who fight against what they do not understand, run some risks; they are told that they ought to comprehend; they do not succeed in doing so, and regard themselves as imbeciles, while this epithet really belongs to those who make themselves believe that they understand." — J. Tannery, *In Padé, Algèbre*, Paris, 1892, p. vii.

"Towards the whole truth with all our heart: on it, no, because it is a meaningless requirement." — Dewey, The Psychological and Logical in Geometry, *Educ. Rev.* 1903, pp. 387–399.

to fill up before completing the deductive edifice, but whatever is done will be well done, and progress can be constructive only, free from the need of correcting the faults and bad habits of earlier days.

It is of the utmost importance that the teacher himself have clear ideas as to rigor of proof, and these ideas are perhaps best acquired by a somewhat extensive study of mathematics, even of subjects not apparently related to those of the elementary field.[8]

Generalization on Insufficient Data.

Extension of proofs. Generalization on insufficient data should be carefully avoided. Thus when in algebra the idea of number is extended so as to include negative number, it does not follow that the theorems which have been proved for arithmetical numbers will necessarily hold also for algebraic numbers. The proofs were made without thought of algebraic numbers, and the reasons alleged for arithmetical numbers, and which were true for that class of numbers, may not be true for algebraic numbers.

How a Proof may be extended.

To establish that a proof made for certain definitions holds for more general definitions not contemplated when the proof was made, it is necessary to go over the proof and ascertain whether or not all the reasons alleged are valid also for the more general definitions, even though these were not considered in making the first proof. If so, the generalized proposition also holds. This is usually the case when the propositions of arithmetic are generalized to cover algebraic numbers. It is not the case when the binomial theorem for positive integral exponents is generalized to include fractional exponents. The reasons alleged in the proof for positive integral exponents no longer hold when applied to fractional exponents (in fact they then lose their sense), and it is necessary to devise a new proof.

The conditions under which a proof may be extended can be expressed as follows:

[8] See Chapter IX., On the preparation of teachers.

Suppose a proposition P to be proved by use of certain definitions or objects of thought D_1 D_2 . . . D_l, certain axioms A_1 A_2 . . . A_n, and certain propositions previously proved P_1 P_2 . . . P_r, and of no others; if now the definitions are changed into D_1' D_2' . . . D_l', and if for these definitions the axioms and propositions become A_1' A_2' . . . A_n', P_1' P_2' . . . P_r', and are valid,—then the proposition P' which results from stating proposition P in terms of the new definitions is also true.

As example, we may take the theorem:

An Example.

Given four points A, B, C, D, no three of which are collinear; if the straight line determined by the points A, B and that determined by the points C, D pass through a common point, then the four given points lie in one plane.

Proof.	*Reasons.*
Determine the plane passing through the points A, B, C.	Three noncollinear points determine a plane.
The line determined by the points A, B lies in this plane.	Every point lying on a straight line lies in a plane, if any two of the points lying on the line lie in the plane.
Therefore the common point of the lines AB and CD lies in the plane.	For it is a point lying on the line AB.
Hence the line CD lies in this plane.	For two of its points (the point C and the common point) lie in that plane.
Therefore the point D lies in the plane.	For it is a point lying on line CD.

If now we change "points" into "planes" and *vice versa*, "lying on" or "lying in" into "passing through" and *vice versa*, "noncollinear" or "no three of which are collinear" into "intersecting in lines of which no two are parallel," and make the necessary grammatical changes, we see on examination (but examination is necessary) that every reason alleged is true also in the changed form. Therefore the proposition resulting from making the same changes in the original proposition must also be true, namely:

Given four planes A, B, C, D, intersecting in lines no two of which are parallel; if the straight line determined by the planes A, B and that determined by the planes C, D lie in a common plane, then the four planes pass through one point.

An Important Distinction.

Truths vs. *conventions.* There is an important distinction between truths and conventions, which the teacher needs always to bear in mind, and to inculcate as occasion may arise. To illustrate:

"According as the subtraction or the multiplication is performed first, the expression 24 − 3 × 5, has the value 105 or 9. What is its true value?"

Reply: "It has no true value. An agreement as to order of operations is required. The agreement ordinarily made is that multiplications take priority, but it would be just as natural, just as 'true,' to agree to perform the operations in order of reading from left to right."

The view and teaching are far too prevalent, that there is a "true" value inherent in every combination of mathematical symbols (apart from the conventions as to the meaning of their combination) and the tacit assumption that combinations defined for some quantities hold in consequence for others not considered in the definition. This gives color to objections like that of Poe.[4] The thought must be clear and

[4] "In short, I never yet encountered the mere mathematician who could be trusted out of equal roots, or one who did not clandestinely hold it as a point of his faith that $x^2 + px$ was absolutely and unconditionally equal to q. Say to one of these gentlemen by way of experiment, if you please, that you believe that occasions may occur where $x^2 + px$ is not altogether equal to q, and, having made him understand what you mean, get out of his reach as speedily as convenient, for, beyond doubt, he will endeavor to knock you down." — Poe, *Purloined Letter*.

potent in the teacher's mind from the very beginning that mathematical symbols are convenient arbitrary conventions, not eternal verities.

The simple steps of mathematics; degree of thoroughness of work. The importance of cutting work up into simple steps and taking them *one at a time*, cannot be over-emphasized. All mathematics consists of combinations of simple and easy steps. The most complicated work is but a succession of such steps. When the single steps are understood, their combinations, by twos, by threes, from the simple to the sufficiently complex should be taken up. When making the combinations the single steps should no longer offer difficulty; the difficulty now lies in the combination, in the complexity.

One Step at a Time.

By thus separating the difficulties and vanquishing them one at a time, the most complex mathematics needed in the secondary curriculum can be conquered with ease, for every single step is in itself easily evident, and thus the simplicity of mathematics which has been repeatedly mentioned, is brought into evidence. As a matter of fact, the majority of pupils do not find mathematics simple or easily evident: for many it is the type of all that is complex, obscure, and hard to understand. This does not point to any inherent lack of simplicity in the subject matter, but to a disregard of its real simplicity in the presentation and study.

Perhaps the lack of success in the appreciation of the clearness and simplicity of mathematics can be ascribed, more than to any other single cause, to the pupil's proceeding to build on what he has not first firmly fixed: the whole edifice becomes insecure; the higher he builds, the more it topples, and the end is complete collapse, perhaps due to a single insecure stone. Troubles in mathematics must not be allowed to rest. Spontaneous cure is very rare. When difficulties arise, one must at once go back far enough to come to solid ground, then rebuild, step by step, on that.

On the other hand, what has just been said may easily be pushed to an extreme.[5] There is danger that pupils, especially when working alone, may sometimes delay too long on same topic, in the attempt to do thorough work, to leave no step unmastered, or that the teacher with the same aim may dwell too long on topics understood sufficiently for the occasion.[6] The distinction must be borne in mind between scientific mastery of a topic and sufficient mastery of it for its next uses: topics on which nothing further is to be built require less thorough treatment than those on which others are to rest. Even such topics may often be laid aside for a time. It is frequent recurrence to a topic after intervals of "unconscious cerebration" and general growth that gives mastery.[7] The important thing is not to build on insecure foundations.

Danger of too long Delay on same Topic.

In developing a new idea or method the examples selected should be very simple, involving, so far as possible, only such subsidiary processes and calculations as have by considerable practice become automatic, so that the management of the

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[5] See Chrystal, *Algebra*, Vol. 2, p. viii.

[6] "The practice of dwelling unnecessarily long on things familiar and essentially simple inculcates a habit of pottering, and is quite as likely to result in confusion of ideas as in lucidity. The fact that a pupil cannot give a clear account of some particular fact or law is no sure proof that he has not spent too much time on it. It is possible to gaze at one's own name until it looks unfamiliar and weird.

"Movement, a certain sense of progress, is essential to the best working of the pupil's mind, which, like a bicycle, simply lies down if kept too long in one spot. It is better to maintain this progress even with the certainty that some things will be passed unseen, and that many of the things seen will be forgotten, than to lose headway and the alertness which goes with it. Many repetitions are necessary for the mastery of certain truths; but these repetitions should not all come at one stretch. An occasional return to the difficult point when the mind is fresh is better in many cases than the attempt to level every obstacle and clear up every doubt at the first progress." — Hall, *Physics*, pp. 319, 320.

[7] See Carpenter, *Mental Physiology*, London, 1881, pp. 536–539.

machinery of the example is practically unconscious, and all the attention can be directed to the new idea involved.

Complexity to be avoided. Mere complexity for the sake of complexity to be avoided always. The need in practice is a safe criterion. If pupil can extract $\sqrt{2}$ to 4 dec. places he can extract it to 40; but it would be a waste of energy to have him do it. True, it gives some drill in operation with integers; but this drill can be obtained just as well in other and more profitable connections.

On the other hand, it may be profitable to have pupils compute the natural logarithms of a few numbers (say 2, 3, 7, 43) to 8 or 10 decimal places. To the pupil at this stage of mathematical work the computation furnishes a concrete application of the formula, an experimental illustration of its elegance and power, and gives incidental review of operations of arithmetic as well.

Style in Mathematics. *Mathematics as a language.* Mathematics has a language of its own. Every equation, inequality, etc., is a sentence. The verbs are $=$, $>$, $<$, etc. A mathematical discussion is a composition in mathematical language, and in all written work the aim should be to have good composition; not only correct statements, but statements in accord with mathematical orthography and grammar. The work should not consist of disjointed clauses, but of complete and clear mathematical sentences, so that any one versed in the language may read and comprehend with the same ease with which he would get the idea to be conveyed in an English composition. This is essential. But how many pieces of written work comply with it? How many are free from false but well-meant statements, like $8 + 6 = 14 + 3 = 17$? How many are free from nonsensical statements? How many tell what the writer wishes to tell without a word of supplementary explanation on his part?

No lower standard than that of ordinary correctness of form, of expression of meaning so that others can understand it, can be set for the language of mathematics. Beyond this, elegance of style is an ideal to be held up, but the degree in which it is attained depends, as in other languages, upon the talents and environment of the individual.

The teacher should practise accuracy of expression always, and *cultivate* it in his pupils — not expecting them already to have attained it, but steadily to be progressing towards it. This is especially true in the first parts of subjects and topics, when harm may readily be done by diverting attention from the new and perhaps elusive thought to the form in which it is expressed.

The Language of Mathematics a Shorthand.

As the pupil acquires the mathematical vocabulary, he should *use* it. Some pupils have too great a tendency to write things out in full. The symbolism of mathematics is a shorthand, and has marked advantages; though the pupil must always know the meaning of his shorthand and be able to translate it back into its unabridged expression. The mathematical shorthand adds nothing to the meaning, it simply records the meaning in a form more perspicuous to the eye. If the meaning behind the symbol is lost, it is worse than useless to continue to juggle with the symbol. The mathematical symbolism must always record or express thought. The symbols are tools, not objects of thought; and the feverish haste of some pupils to be by all means writing something in the mathematical language, whether or not they have anything to say, should be discouraged emphatically at all times.

English in Mathematics.

It goes without saying that the English used in mathematics should be good, and that by precept, and especially by example, the teacher should always inculcate the use of good English.

Problems should be real to pupil.

Character of problems. A problem assigned for work should be a problem *for the pupil*, one which he recognizes as needing solution, permitting solution, and deserving solution. In no way can this end be achieved better than when the problem relates in the broadest sense to the pupil's own activity, when they confront him in the course of what he himself is doing. It may, however, not be practicable to choose all problems so; the next best thing is to choose problems which the pupil knows are problems for some one, even though they may not be confronting him at the time. Problems which bear the stamp of pure artificiality on their face lack an important element. True, many of the ends of the study of mathematics can be attained by their use, but there is such an abundance of real, live, interesting problems that *all* the ends can be attained without their use.

Problems should be definite.

Problems should always be stated definitely, so that there can be no question as to proper result. For example, "factor $12\,x^4 - 48\,a^2 x^2$" is not definite. It can be done in numerous ways even without admitting fractional or irrational factors. An indefinite result is worthless, and an indefinite problem is worse than no problem.

The Purpose of Home Work.

Home work. The chief purposes of work assigned for study by the pupil apart from the teacher are:

1. Drill on operations whose theory is understood.
2. To impress on the memory those few things which need to be memorized.
3. To inculcate neatness.
4. To give opportunity for quiet thinking.

The most effective home work is that which has the character of completing the class work of the previous day, not of preparing for the next.

What should be assigned.

It is not advisable to assign work unless it has been sufficiently developed in the class to enable even the dull pupil to apply his time to it with success and profit. The pupil should never be set to struggle with really new matter without the supervision of the teacher. The work assigned may require thinking, problems in some respects new may be given for solution, but there should always have been enough similar antecedent work to furnish pupils a clue sufficient to prevent their working in the dark.[8]

[8] The chapter on "The Philosophy of the Assignment," in Carpenter, Baker, and Scott, *The Teaching of English*, New York, 1903, pp. 319–326, is suggestive.

With the writer, the plan has worked excellently of having each member of the class hand in a written report of his home work at the beginning of the next class exercise.

How treated afterwards.

The reports are made on uniform[9] cards and the work assigned is classified under three heads: *Satisfactorily done*, *tried without satisfactory results*, and *not tried*. Everything assigned to be done is reported by number on the one or the other of these heads, and with this information in hand the teacher can most briefly and effectively take up the points of difficulty without losing time on those which have given no trouble; this expeditious mode of locating and meeting the difficulties generally permits the teacher to give the larger part of the class period to the new work. The fact that a pupil may be asked to explain work which he has done satisfactorily, for the benefit of those who have not, serves as check upon incorrect reports. This daily written report, on a uniform blank and in a uniform notation, enables the teacher to get the content of a report at a glance, and is thus a simple and very effective device, by means of which the teacher with almost no effort or loss of time can keep in constant touch with the work of each pupil. The "not tried" column sounds the warning note if the assignments are by inadvertence too heavy. The space for remarks may be utilized whenever the pupil desires, results may be reported, short written exercises may be prepared on the back of the card, pupils may be asked to make problems themselves bearing on the topic in hand, and these problems, written on the back of the cards, may be distributed to other pupils for solution; still other uses of the cards will suggest themselves to the teacher in practice. The use of the cards also obviates the need of roll-call, and thus saves quite a little time.

Whether by such reports or in some other way the teacher should somehow ascertain and clear up the difficulties met in the home work; this will usually not require much time,

[9] Published by A. Flanagan, Chicago.

leaving the bulk of the class period free for the work in advance. "Not prepared" should have *no sense* during at least the larger part of the time, for the questions should call for present thinking rather than for evidence of past preparation.

Occasional papers prepared with care at home, marked by the teacher, and returned, are desirable.

It is very helpful for the teacher to keep for himself a record of what is done each day, the topics taken up, the assignment for next time, the things to do, weak points to strengthen in the general work of the class and in that of individuals, and that he check off these desiderata as accomplished.

Teacher's Record.

Exhibition of essentials. Every subject consists of a few essentials with many ramifications. It may be a good thing to have the pupils (in class, with the teacher in the lead) prepare an outline of the principal results already attained. After class discussion this skeleton should be neatly written out, enlarged from time to time, and made the basis of drill. Such a syllabus will not only fix the results themselves in mind, but will give the pupils a much better appreciation than they usually have of the proportions of the subject, the relations of its parts, its perspective.

Syllabus.

Mode of explanations at blackboard. A minor point apparently, and yet one that contributes much to best results in the class exercise, is the constant practice on the part of both teacher and pupils of writing and simultaneously repeating orally or explaining what is written. This is undoubtedly more difficult than first to write out a demonstration and afterwards to read it and explain it. It requires three simultaneous activities, — thinking, writing, speaking, — but with practice from the earliest years the habit can be well established to the great advantage of the mathematical class exercise. In Germany pupils "chalk and talk" with ease from the beginning. Quite a number of advantages are gained by this procedure:

"Chalk and Talk."

1. The figure or proof is developed as needed and as explained. Earlier steps are not encumbered by what is not

needed until later. The *growth* of the proof is secured; the life is attained which a text cannot have, and which is largely lost if the blackboard is treated simply as a place where an enlarged copy of the text may be brought before the eyes of all.

2. The attention of all pupils is directed to the same point of the proof at the same time.

3. A strong barrier to mechanical work lies in the fact that the pupil must explain each step as he takes it. Unforeseen discussion may take place, making the path of the mere memorizer much harder than if he is given the opportunity quietly to write out what he has memorized and afterwards to read off the whole.

This Ability: how attained.

That the process is difficult cannot be denied, yet if the pupil thinks out his results he must do something of the same sort in his private work; the requisite skill can be secured only by long practice, and the use of this mode of explanation from the very beginning of the study of mathematics. The act of writing should become semi-automatic, as it is in writing out one's thoughts in ordinary English. In the case of long or complicated explanations, or whenever needed, the pupil may be allowed to refer to his written notes, not to copy from them *verbatim*, but to aid the memory in recalling the next step. It is bad economy to waste the time of a whole class while one pupil gropes about for lack of a little prompting.

Remember, not memorize.

The rôle of memory in mathematics. The function of memory in mathematics has undoubtedly been overemphasized. There is no question that there has been mathematical teaching which required simply memorizing and reciting the words of some text-book, and there is likewise no question that such teaching is thoroughly bad. On the other hand, the more quickly and exactly previous results are remembered the more rapid will be the progress. It is highly desirable to *remember* results previously understood: it is highly pernicious merely to *memorize* the words of others, perhaps not even well understood.

If the pupil is required to do little, if any, memorizing in mathematics, but permitted to look up freely any or all of his previous results whenever he needs them, memorizing will be supplanted by remembering. By dint of repeated lookings up he will begin to remember the most important results, and by and by he will see that there a few seconds or minutes spent in fixing definitely in the memory what is already almost remembered would be time well spent, and obviate a great deal of looking up after. Memorizing will thus come *after* the result has been attained and used somewhat: it is the climax of remembering.

Though seldom requiring memorizing the teacher will, of course, point out the results most to be used thereafter and which it would be most convenient to be able to remember accurately and readily, and what memorizing is done becomes simply a gathering together and organizing of known results already remembered in large part. In this way remembering and memorizing are reduced to their proper status in mathematical instruction, — that of useful, indispensable auxiliaries, not that of the prevailing mould in which all the work must be cast.

The aim of the work is to make the pupil master of the thought content and to enable him to apply it. Whether or not he is able to get up before a class and reproduce it *memoriter* is of small consequence. It is sufficient that he shows by his presentation and replies to questions that he has grasped the thought. The mere consciousness of the presence of a prompter in case of need will remove the nervous strain of fear that he may not "remember what comes next," and diminish the need that the prompter give the cue. It is pathetic to see an embarrassed and dejected pupil struggling, mathematics becoming hideous to him, simply because his memory plays him false. After an experience or two of this sort, the very fact of being "called on" may suffice to induce real "stage fright." Freed from the tyranny of memorizing, all the energies of the pupil are bent on thinking, and what is once clearly thought through is already half remembered.

Marking system. Whether a teacher should make a daily record of the quality of the work done by each pupil, and, if so, what degree of precision should be given to the record, are questions which will receive various replies from different teachers.[10] At all events, whatever marking system may be used should be a servant, not a master. Some record of the pupil's work seems desirable. With perhaps one hundred or more pupils, many of them perfect strangers at the outset, the teacher can hardly carry in mind the work and needs of each without the assistance of some written record. A physician would be considered culpable if he carried along the treatments of one hundred patients with no written record whatever of any of them. With a suitably selected notation the teacher can readily record his daily impression of the quality of the pupil's work. The beginner in teaching, at least, should certainly make such a record: if after considerable experience he finds that less frequent entries accomplish the purpose, he will diminish their number. The record is an aid to the teacher, and not a master deciding by tenths or hundredths, apart from the teacher's general judgment, whether or not the pupil shall receive credit for the work. The system of a weekly or monthly record of teacher's grades in a central office where the average made by a clerk at the end of the year decides the pupil's fate is not at all to be commended.

Some Record needed.

An excellent form of record is one, modelled somewhat on a physician's record, in which a page large enough to do for the entire course is set apart for each pupil. The teacher looks over the record each day immediately after the close of the class exercise, and makes such entries in words as may be requisite, — not necessarily an entry every day; for example:

John Smith.

Oct. 3d. Seems uninterested, but understands the work.
4th. The work may be uninteresting because too easy.
Memo. Give him something special to do.
5th. Assigned . . . . for outside work.

[10] See Baldwin, *School Management*, pp. 189–195.

Oct. 7th. He reported work done: handed in solution — Good: commended.
8th. Assigned . . . .
11th. Worked assignment. Seems to take pride in it. *Memo.* Continue.

Such a record, used alone or supplementing a daily grade, may give a good view of each case and be of much help to the teacher.

Economize Energy and Time.

Avoid waste. A waste of one minute's time in the class exercise means a waste of half an hour or more for the totality of the pupils. A certain railway line has, under the writer's own observation, been straightening its lines for the last twenty years, here a little, there a little, at a great expense for what seems a trifling improvement, and yet the total of all the resultant trifling economies, each repeated many times daily, year in and year out, has evidently proved a satisfactory return for the enormous outlay, for the same policy is being continued. So in the mathematical class-room. Let all the curves of routine be made as smooth and easy as possible. Let the mechanical movements of the class, and of single pupils, to and from the boards, etc., always be made promptly in the same way and with the same word of direction. The board should be cleaned before each class. If the janitor is not available for this work, an arrangement should be made whereby the pupils do it. Whatever time is spent on mechanical things in the class, beyond the irreducible minimum, is a direct waste.

Single pupil mode. By the "single pupil mode" is meant the mode in which one pupil is called to his feet and the class exercise devoted to him for a considerable length of time, either by letting him give an uninterrupted explanation or by a dialogue between him and the teacher. There may be an occasional question directed to some other member of the class, but essentially the one pupil has the floor and the others are spectators.

I have seen good teachers using this mode with success, and there are occasions when perhaps it gives the best results;

but basing my views upon visits to many classes, in many schools, over a wide range of territory, I fear that in the United States the mode is used much more than it should be. Whatever puts the pupils in the passive attitude is, by that very fact, questionable. In general every one of them should be actively at work — not necessarily with his hands — always with his mind. When a single pupil is given the floor, it should be for the benefit of the class rather than his own. Occasionally, of course, a good pupil may explain some knotty point as well as the teacher, and it is preferable to have it so explained. In general, however, the teacher is the single person most likely to occupy the time of the class to best advantage, and unless the time is so occupied it is wrongly occupied.

Passive Attitude of Pupils to be avoided.

The special disadvantages of the single pupil mode are that inattention on the part of the others is made easy, is almost at a premium, that the cardinal principles of active work and united work by all are temporarily suspended, and that the exercise offers strong temptations to make it mere cross-examination of the single pupil.

In this connection the question arises as to who should do the work at the board when a topic is being discussed by the class. The general rule would seem to be that the pupil should do it, exception being made of work in which it is not possible for the teacher so to direct the pupil's steps as to prevent him from going astray or losing time. These cases will probably not be numerous. Only pupils who can do what is desired should be allowed thus to work singly at the board. A pupil who does not see what is wanted of him as rapidly as the majority of the class see it should be replaced by another. However beneficial the drill may be to him, the time of the whole class should not be squandered for the benefit of one pupil. If the difficulty is typical, the matter is of course quite different.

When should Teacher work at Board and when Pupil?

With other pupils, of advanced high school and collegiate grade, whose power of attention is stronger, the single pupil mode may be used more than in the earlier years, particularly for the presentation of the results of personal assignments made to the different members of the class. Even in earlier years single pupils may be given opportunity to show work of special merit which they have accomplished. This is vastly different from trying to teach the class over the head of a single pupil, but still the single pupil mode may well occupy, even in the more advanced years, only a minor portion of the class time.

Special Assignments.

The blackboard mode. A mode much used, and one of the highest value, is that of having all the pupils working at the board simultaneously, on problems dictated by the teacher. A brief exercise of this sort may well follow any development of some theoretic point, and it may also be used to develop new thoughts. It shows the teacher at once who has understood the theory and who has not, and enables him to give a series of easy problems tending to remedy the defect, — the exercise beginning with very simple problems, surely within the reach of all, and increasing with difficulty until the desired facility has been attained. The direct view which the teacher has of every pupil's work, and of every pupil at work, enables him to carry the exercise to just the right point, continuing long enough, but not too long. As soon as the teacher sees that the pupils have grasped the central idea, and need only practice, similar exercises for home work may be assigned, the class returning expeditiously to their seats, refreshed by the physical change of working at the board, and ready for further discussion of theory. The careful teacher will develop but little theory at a time, and will at once assure himself by some practical test that the pupils have grasped it and are able to apply it. When in exceptional cases the teacher finds it best to give some matter didactically, he will immediately call for its return from the member of the class to assure himself that they have seized the meaning as he wishes them to.

Advantages of the Mode.

Extemporizing problems. Such exercises frequently require the teacher to extemporize problems — and this is an art in itself. A poor problem, one that is trivial or beside the point, is worse than none. The best way to learn to extemporize is — not to extemporize, but to prepare every set of problems carefully, studying the topic from all points of view and trying to be prepared for all contingencies. The material thus stored up may be used later, and such extemporizing, the unexpected use of material previously prepared, is the only extemporizing that the teacher should permit himself to undertake. Thus, extemporizing is not a thing to be attained or sought directly, but is a by-product of much and thorough preparation.

How to extemporize.

Making problems. The teacher cannot rely upon the text to furnish all the problems. It may properly furnish many, including the more difficult ones, but the teacher will always find need to supplement it with special problems to meet special needs, and, most of all, with interesting problems based on local conditions. The latter are discussed in other connections. At this point a word may be added with respect to abstract problems.

Source of Problems.

To make such problems effectively, requires on the part of the teacher a thorough mastery of the topic in its various aspects. The more general points of view of later mathematics are often suggestive, and the general formulæ there attained are in many cases moulds in which particular numerical problems for elementary mathematics can be cast with ease and rapidity. The elementary theory of algebraic equations [11] is a rich mine of such material, which, duly digested, will enable the teacher with little trouble to prepare an abundance of problems suitable for elementary algebra.

As a more detailed example consider the relation:

$$(m^2 + r^2)^2 = m^4 + 2\,m^2 r^2 + r^4$$
$$= (m^2 - r^2)^2 + 4\,m^2 r^2.$$

By putting

$$m^2 + r^2 = c$$
$$m^2 - r^2 = a$$
$$2\,m\,r = b,$$

this relation becomes $c^2 = a^2 + b^2$;

[11] As found, for example, in Burnside & Panton's *Theory of Equations*, 4th ed., London, 1899.

that is, for all positive values of m and r, $m > r$, a, b, c, are sides of right triangles and are integral whenever m and r are integers.

By using the value of a, b, c, given by these formulæ, numerical values *ad libitum* can be obtained for problems concerning right triangles in which the teacher knows, without making the calculation himself, that the square roots arising can be extracted exactly and that the results will be integers.

Reviews. That reviews are of the utmost importance, that only by recurring to it again and again can a topic be mastered, is a pedagogic banality. Our advertisers know that constant repetition fixes an idea, otherwise why the thousands of dollars spent in demanding, "Have you used X's soap?"

Reviews may be either direct or incidental. The former need not merely retraverse exactly the same ground in exactly the same way; fundamentals must be retraced again and again from different points of view; miscellaneous exercises should also be taken up, involving only what has already been mastered, but without particular order and without hint as to what process is to be used; — giving training in deciding what ones to use of the many processes already had, and developing a most important power, the ability to apply what has been learned, in which pupils are often found deficient, both by physicists and in the business world.

Direct and Incidental Reviews.

Incidentally processes are reviewed whenever they are used in building up something else. This is perhaps a still more important way to review, not laying a process aside after its theory has once been explained and practised, but keeping it fresh in mind and making it more and more the real mental property of the pupil by frequent incidental use.

Teaching through the eye. It is not only a matter of common belief based on ordinary experience, but well established by exact experiments, that *seeing* is to most minds more effective than hearing, and that hence it is better to teach through the eye than merely through the ear.

Several instrumentalities are especially useful in doing this:

1. *Neatness and orderly arrangement of work.* In American schools especially, too much stress cannot be laid on this point. Our children are far behind French and German children in this respect. When students who have received secondary school training in France or Germany enter American classes, the neatness of their work alone usually tells the instructor from what schools they come.

The mathematical teacher can hardly remedy the matter alone, but he can contribute his part. Neat work, good writing, should be insisted on by *all* teachers, and slovenly papers rejected unhesitatingly by all. Only by insistence from the earliest years can the habit of neatness be established. Once thoroughly established, it persists like any other habit.

"The habit of careless and slovenly work once acquired is very difficult to cure, and it leads to a state of mind which is very hurtful in things other than mathematical."[12]

2. *Squared papers.* This unrivalled instrument for exhibiting many mathematical relationships to the eye has been discussed elsewhere (Chapter VI) and need be only mentioned here.

3. *Use of colors.* Difference of color can be utilized in many ways to make things clearer to the eye: equal angles or lines may be marked with the same color; known parts may be marked with one color, parts sought with another; subsidiary or construction lines may be in a different color still; etc.

A few different colors will suffice; in most cases two. Colors should be used which stand out clearly on the board (or paper) and which contrast well with each other. The best in order are white (black on paper), bright red, bright yellow (on board; blue on paper), bright green. Only the desired colors should be procured. Boxes containing miscellaneous assortments are to be avoided, as they contain much that cannot be used.

4. *Schematic analyses.* The various analyses, syllabi, etc., which may be prepared in the class, as has been suggested in

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[12] Mathews, *l. c.*, p. 191.

other connections, gain much of their effectiveness from a schematic arrangement that makes the relationship between the different parts stand out clearly to the eye.

Free questioning. Free questioning on the part of the pupils is by all means to be encouraged, even though it may now and again necessitate an "I don't know" on the part of the teacher. The teacher's prestige should be so well established by positive knowledge shown that he need have no fear of an occasional "I don't know." Let him find out if possible. The pupil will feel that he has really thought of something worth while if even the teacher does not know the reply, and he will be much keener to find or learn the result.

The questions should be specific (perhaps in writing), the answers succinct and to the point; often in the form of a hint or counter question. The teacher need not hesitate to say, "I can give you a clearer (better) answer to-morrow."

In permitting free questioning by the pupils, the danger arises that some pupils may try to get the teacher to talk to avoid doing work themselves. The teacher will of course quietly be on his guard against this, and will not permit himself to take up time in discussing irrelevant questions, or those which do not fit in well into the programme for the class period which he has outlined in his preparation. This danger can easily be avoided. As a rule the questions are sufficiently genuine and unsophisticated, and in many cases the teacher can attain his own goal while apparently simply following the lead of the class.

The use of a text. A number of reasons make the use of a text very desirable. It prevents useless dictation and taking

Why a Text should be used.

of notes which are likely to be imperfect, it furnishes a good collection of exercises prepared with great care and at the cost of much more time and pains than any single teacher could possibly give; it places within the reach of the pupil a theoretic development worked out with much thought and a wide view of the bearings of the subject.

In the use of a text what should be the teacher's attitude towards it? Should the teacher be a commentator on the text, or the text a supplement to the teacher? Ideally, these questions admit of but one answer, and the ideal is by no means out of the reach of the average teacher. With the teacher who has attained due mastery of his subject, the text-book, though wellnigh indispensable, is but a tool. Except for strong reasons he adheres to its general order and spirit, though he does not regard it as a semi-inspired piece of work neither to be added to nor diminished from. Ideally, its chief functions are those of a work of reference and a collection of exercises. In the class discussions other solutions than that given in the text may be suggested by pupils. Such solutions are, of course, to be welcomed and worked out when practicable, though the teacher will usually see to it that that of the text is also outlined — perhaps compared with the others. Even though that of the text may be simpler, the pupil should be led to feel that the chief thing is to find *some* correct solution, and that he has no reason to feel dissatisfied because he did not find the simplest or best at first. It is seldom the case that a genuine discovery or invention does not admit of considerable improvement. An interesting illustration for class use is the proof of the transcendency of the number π, which establishes the impossibility of "squaring the circle."[18] Well-known inventions — telephone, phonograph, bicycle, automobile — will furnish other illustrations.

How to use the Text.

In how far the teacher may well vary from the text depends upon many circumstances. In general, the presumption is that the mode of presentation of the text will be followed, but that variations will be made for good reasons. Does variation from the text necessarily imply that the teacher believes that he knows a better presentation? Not always. The fundamental reason is the craving to establish *personal* thought-circuits between teacher and pupil, — a live current, unbroken by the dead text. When the teacher has made some domain of mathematics thoroughly his own, he feels

Varying from Text.

[18] See Klein's *Famous Problem of Antiquity*, transl. Beman & Smith, Boston, 1897.

that the proofs, even though exactly like those of some text, come more effectively in direct personal relations between himself and his class without thought at the time of any third party.

One may learn streets from a map, and the map furnishes the first guide in going about. But after one knows the streets thoroughly, he can tell another the way on his own authority, without citing the map or thinking of it.

Note-books. The function of a note-book may vary widely with the mode of instruction used and with the age of the pupil. The value increases as the pupil grows more mature and the work becomes more formal. In Germany simultaneous work in note-books takes the place of blackboard work. Whatever may be done by the teacher or a pupil at the board is done by all the other pupils in the books at the same time, so that at the close of the hour the books contain a record of the work of the day as the pupil has understood it. Weak points of the practice as seen there are that the pupil is not required to rewrite the matter in better and more complete form, and that the teacher at no time inspects the note-books.

A good suggestion is that of "day book and ledger," [14] — the first for rough notes as the work develops, the second for systematic permanent record.

When a good text is in the hands of the pupils, it would not seem to be necessary for the pupil to make note of those things which are to be found in the text. One of the main uses to which a good text may be put is that of subsequent reference to clear up what may not have been thoroughly grasped in the class exercise. The pupil's own note-book would be of little help here, since his notes would be imperfect or incorrect on just these points.

Chief Uses of the Note-book. Uses of the note-book are: 1. For rough work in class; 2. To record theoretic developments made in class and not found in the text; 3. To record the assignments.

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[14] Osborn, C. S. Thought Values in beginning Algebra. *Sch. Rev.* 1902.

When matters are being developed orally in the class, the attention should not be distracted by taking notes; after the topic is somewhat formulated, it may be briefly recorded as far as necessary to supplement the text. Unless fixed by notes, the material is likely to be lost.

Ends of Written Exercises.

Written exercises. The value of written exercises is evident as giving the teacher opportunity to see to what extent the pupil has the work in hand, and what points are still weak; of training the pupil to quiet thinking, accurate work, clearness of style, orderly arrangement, to neatness, to careful expression. But the written exercises must be so conducted as to *attain* these ends. Almost all of them are often missed by an assignment too large for the allotted time. The result is work that is slovenly and careless in form and in spirit, the very opposite of what is desired. Here, if anywhere, the motto should be "a little, but thoroughly well done."

The written exercises may either be a class-room test or work assigned for private preparation. The class-room test may either be announced beforehand and its general scope specified, or it may be given unexpectedly. All these types are good; especially valuable is the frequent short unexpected written exercise of from five to fifteen minutes at the close of the class period, the work required being so little that all the pupils can do it well in the time allotted. Such exercises give variety to the class work, attain many of the ends of written exercises, and are easily examined by the teacher. Specially suitable for home work are assignments requiring systematic synopses of the work of the class, and the careful solution of longer problems which give training in connected presentation; one problem will usually be an ample assignment.

Exercise Books.

The written exercises may all be recorded in the same book. If so, it is a good plan to have the original work confined to the right pages, with a margin of an inch or more blank for the teacher's comments, work not sufficiently well done to be repeated opposite the original, so that the error and its correction can easily be compared.

To attain the full benefits of written exercises, it seems indispensable that they be marked by the teacher and returned to the pupil. Even with all the devices of abridged notations, the marking requires an almost impossible expenditure of time on the part of the teacher.

Marking of Exercises.

The young teacher should undergo this work willingly, for it is the best of pedagogic training. In no other way is he brought so close to the real state of affairs with his pupils, to their different needs, their individualities: many and many a time will he be surprised when the written exercises show a startling lack of comprehension of what he thought had been made so clear in the oral work. For quite a number of years the teacher should welcome the marking of papers as an instructive privilege: after a decade of such experience he may begin to cast about for time-saving devices, but rarely will he find any. The only way to mark papers is to mark them.

Note the expression is *mark* papers, not *correct* papers. The errors should not as a rule be corrected, but marked, with sufficient hint to enable the pupil to proceed intelligently to correct the error. The correction itself should be examined by the teacher, remarked and recorrected as often as necessary. It is not necessary to correct all the faults at once; let the pupils understand that no mark is not necessarily equivalent to the teacher's endorsement. Perfection must be attained gradually. When the most egregious faults have been overcome, the minor ones can be taken up. The teacher will do well to note in his record what are the most serious faults of each pupil.

If papers are given a grade, it should be on principles understood by the pupils in the same way as by the teacher. It is well to grade only generally, — excellent, good, satisfactory, unsatisfactory, poor, or the like; it is impossible to estimate work justly to a fraction of one per cent, and comparison of such fictitiously accurate grades causes unnecessary heartburnings.

Grading.

Transcriptions should be reduced to a minimum. Let the ideal be that the first work should be good enough to keep, to hand in. It is better, however, to throw away than to patch up work.

Both teachers and pupils should understand that a pupil may often show as much or more mathematical ability in a problem that he fails to solve as in one that he solves.

The custom of marking a problem zero if it falls short in any particular whatever, fails to realize the true significance of problem solving.

A few words of counsel fitting the individual case will naturally be written on each paper where needed: matter of a more general character, errors common to a number of papers, etc., may with profit be briefly discussed before the whole class.

Examinations — Two Sorts.

Examinations. Examinations may have two widely different characters: one, as a test of the pupil's attainments by some outside authority and in accordance with some oustide standard; the other, as the culminating class exercise.

In the first sense examinations may be regarded as necessary evils and their influence upon instruction as bad. It is needless to discuss examinations from this point of view. In any system in which all or nearly all hinges upon the result of an examination of some outside authority, the examination is a *fact*, to which the teacher is compelled to bend his teaching, and no amount of theorizing will ever lead him to do otherwise. Fortunately, this extreme form of the examination is by no means predominant in the United States; the influence of outside authority is on the whole no more than a healthy stimulus to good work, and the teacher is free to treat his examination in the second sense, as the culmination of the class work.

The Purpose of the Examination. What it is not.

A few words as to what the examination is *not*. It is not (should not be) to any great extent a factor in determining whether or not the pupil is to obtain credit for the course. The pupils should understand that steady good work is what counts, and that while the examination is of some weight, it is not likely to be sufficiently bad to overbalance good work throughout the course, nor sufficiently good to atone for radically defective work. This will go far to shear the examination of its terrors for those timid souls whose very trepidation and fear lest some error should ruin their hopes prevents them from doing themselves justice, and will also give fair warning to those who are inclined to shirk during the course, in the hope that by a little "cramming" at the end they may succeed in passing a good examination and so setting matters straight. The examination is not, to any extent worth mentioning, a test to inform the teacher as to whether or not the pupils have done their work well. He should know this beforehand.

The Purpose of the Examination. What it is.

To some extent the examination gives color to a thorough review of the whole subject and furnishes a fitting climax to the work of the course. Regarded as a test, the view should be towards the future, not towards the past. The pupil is about to quit the formal study of the subject; thereafter, if he uses it, it will be by way of application. The examination, in so far as it is a test, should test his ability to *use* the subject, and the test should be made under the conditions under which he will actually use it. This would permit him free access to text-books, notes, and any other books he may wish to bring to the examination. Such a regulation both puts those at ease who fear that their memory may play them false, and shows the uselessness of "cramming." The problems set will of course take account of the conditions of the examination. They will not call for a mere repetition of what has been learned, but will give the pupil easy and fair opportunity to show whether he can use it.

The examination is also a means of instruction, and this last set of problems may easily be made the most interesting and instructive of the whole course. It is a decided advantage if the time of the examination can be so set that the class will hold one session after the examination to enable the teacher to talk the problems over with the class after he has read the papers.

When the examination has this character, the question of exemption for pupils having attained a certain high grade will not arise. The examination is a privilege, not a penalty.

As such an examination requires thought, care should be taken to allow enough time for the requisite thought, and to err on the side of setting too few rather than too many problems. It is not necessary to cover the whole subject. Pupils are usually desirous to have alternative problems, and it works well in practice to assign them, notwithstanding the time that may be lost in hesitating between this or that.

CHAPTER VIII

PREPARATION OF TEACHERS; MATHEMATICAL CLUBS

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The Preparation of the Teacher

Command of Subject Matter taught.

THE first and fundamental requisite for every teacher is that he have thorough command of the subject matter which he teaches; that he have mastered it so well that he speaks with his own authority; only so can he hope to lead the pupil to the corresponding feeling of independent mastery. One can talk freely only about what he has really made his own. If he has not grasped the truth as distinguished from the form, if he feels bound to transmit unaltered to the pupil what he has received from some "author," he is bound hand and foot in the performance of his functions. Every day of a mathematical teacher's life demands that he vary unessentials to make essentials more evident.

This Command how attained.

This mastery of the subject matter is attained in part by intensive study of the subject taught, in part by extensive study of a wide range of mathematical subjects. In fact, the latter variety of study prepares the way well for the former. Teachers may be tempted to confine their personal reading too closely to phases of what they are teaching. A certain amount of such work is very good, but when carried too far without the illumination of much broader mathematical acquisitions it is apt to be sterile, or to lead to an expenditure of talent and valuable energy upon undue elaboration of relatively unimportant matters.

What Subjects should be taken up.

The ideal would be several years of graduate work in mathematics, but at present this must remain in America simply an ideal, though it has long been in full force in Germany with the best results.[1]

A number of lists have been drawn up and published outlining the *minimum* equipment in mathematical knowledge which under the conditions of to-day the aspirant for admission into the ranks of teachers in American secondary schools should be required to possess.[2] These lists are in substantial agreement. They include (besides thorough courses in the subjects of secondary school instruction themselves, arithmetic, plane and solid geometry, algebra, the elements of trigonometry), good courses in trigonometry (surveying desirable), college algebra, plane analytic geometry, the differential and integral

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[1] The high standard of scientific preparation required in Germany has made it possible to recruit from among the teachers in secondary schools a considerable number of the most brilliant mathematicians of the nineteenth century, as Grassmann, Kummer, Plücker, Steiner, Weierstrass. See Klein in Lexis. W. *Die Reform d. Höh. Schulen i/Pr.*, p. 258, Halle, 1902.

[2] Com. on Coll. Ent. Req. Hanus, Young.

calculus, the theory of equations, including determinants; courses in the history of mathematics, and the elements of analytic mechanics; some work in theoretic and practical physics should also be included.

The Preparation, how attained.

The practice is growing of including a college education among the requisites for eligibility to appointment as teacher in secondary schools. This is in accordance with Resolution III. of the Committee on College Entrance Requirements.[8]

In the stronger colleges it is possible for students looking forward to teaching mathematics to include all the subjects named above as electives in the undergraduate course. A year spent in graduate work in mathematics would of course be more desirable, and some graduate work in a strong institution should be taken by graduates of colleges in which it is not possible to secure the minimum equipment as a part of the undergraduate work.

For Active Teachers.

What has been said relates to candidates for appointment, and not to teachers already in ranks, whose fund of practical experiences, though not acquired in the least costly way, is a treasure of greater value in the class-room than the theoretic knowledge of some more advanced subjects. But there are now available numerous facilities for supplementing this fund of practical experience with additional theoretic knowledge, such as the summer sessions of good colleges and universities, extension and correspondence courses, and the knowledge acquired through any of these channels will be assimilated and utilized to double advantage because of the maturity of view due to practical acquaintance with the field in which it is directly or indirectly to bear fruit.

It may be noted that, notwithstanding the three years of

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[8] "III. Resolved, That the teachers in the secondary schools should be college graduates or have the equivalent of a college education." — *Report*, p. 30.

The Report presents a strong array of the reasons in support of this resolution.

graduate work required of all who would teach in German secondary schools, some German universities are organizing vacation courses for the benefit of teachers already at work. These courses are conducted by leading university men, with the object of helping the teacher to keep abreast of the progress of his subject, and are eagerly welcomed by those for whom they are intended.

Advance in Scientific Knowledge needed.

For every one, whether or not already equipped with the minimum requirements or more, it is essential that he continually enlarge the store, and realize in his own activity the fact that mathematics is a live and growing subject. The facilities just named are available for this purpose, as are also private study and joint work with colleagues.

As suitable subjects to take up there may be named: A fuller study of any of the subjects of the minimum list; also modern synthetic geometry, solid analytic geometry, the theory of functions, the theory of numbers. The subject named last would in many instances be especially interesting, because, while it requires some facility in abstract reasoning, it may be taken up with practically no technical mathematics, is easily amenable to numerical exemplifications, and leads readily to the frontier.[4] It is perhaps the only branch of mathematics where there is any possibility that new and valuable discoveries might be made without an extensive acquaintance with technical mathematics.

English Books for Teachers' Reading.

Books on all these subjects are available in English, though many of the best books are in German and French. In English there may be mentioned:

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4 "The most beautiful theorems of higher arithmetic have this peculiarity, that they are easily discovered by induction, while on the other hand their demonstrations lie in exceeding obscurity and can be ferreted out only by very searching investigations. It is precisely this which gives to higher arithmetic that magic charm which has made it the favorite science of leading mathematicians, not to mention its inexhaustible richness, wherein it so far excels all other parts of pure mathematics." — Gauss.

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For additional reading along the line of the minimum list some of the following may be used. They are not, as a rule, best suited to beginners.

Trigonometry.

Hobson.

Locke. (*The treatise.*) London, 1892.

Loney. Second Edition, Cambridge, 1895.

Algebra.

C. Smith. (*Treatise.*) Fourth Edition, London, 1893.

Chrystal. Edinburgh. Second Edition, 1889.

Analytic Geometry.

Smith. Conic Sections. Seventh Edition, London, 1883.

Salmon. Conic Sections. Sixth Edition, London, 1879.

Lamb. Cambridge, 1897.

Calculus.

Echols. New York, 1902.

Gibson. London, 1901.

Theory of Equations.

Burnside and Panton. Fourth Edition, London, 1899. (Vol. I. is suitable for beginners.)

Dickson. New York, 1904. (An introduction to the theory of groups as applied to algebraic equations.)

Cajori. New York, 1904. (Sketches modern results.)

Determinants.

Scott. Cambridge, 1880.

Determinants for Beginners.

Burnside and Panton. London, 1899.

Hanus. Boston, 1886.

Weld. New York, 1895.

Muir. London, 1885.

Foreign Books for Teachers' Reading.

In German, French, and Italian the number of works on the subjects named above is very great. A few will be cited, which might well serve as introduction to the literature of the subject in the language named. Only more elementary works are mentioned. The teacher who is ready to read more advanced works with profit will have ample opportunities for ascertaining from other sources what is available along any given line.

General Arithmetic.

Tannery, J. Arithmétique. Paris, 1900. (A very valuable work; to be highly commended to any who read French.)

Humbert. Arithmétique. Paris, 1901, Second Edition.

Geometry.

Rausenberger. Die Elementargeometrie des Punktes, der Geraden und der Ebene, systematisch und kritisch behandelt. Leipzig, 1887. (An instructive work. Does not go beyond the elementary field, but is written in the modern spirit.)

Hadamard. Géométrie. Paris, 1898.

Rouche et de Comberrouse. Géométrie. Paris, 1898, Sixth Edition.

Faifofer. Géométrie. (Translated from Italian), Paris, 1903.

Enriques. Questioni di Geometria elementare. Bologna, 1900.

Algebra.

Netto. Elementare Algebra. Leipzig, 1904.

Bourlet. Algèbre. Paris, 1896.

Determinants.

Baltzer. Theorie u. Anwendungen der Determinanten, Fifth Edition, Leipzig, 1887.

Mansion. Éléments de la théorie des determinants. (Sixth Edition, Paris, 1900.)

Solid Analytic Geometry.

Rudio. Analytische Geometrie des Raumes. Third Edition, Leipzig, 1901. A simpler work than that of C. Smith, and more suitable for beginners. There is a companion volume for the plane, viz.:

Ganter und Rudio. Analytische Geometrie der Ebene. Fifth Edition, Leipzig, 1903.

Theory of Equations.

Petersen. Theorie der Algebraischen Gleichungen. Kopenhagen, 1878.

Theory of Numbers.

Dirichlet. Dedekind-Zahlentheorie, Braunschweig. Third Edition, 1879. (This is a standard work and as well suited for a beginner as any.)

Gauss's classic *Disquisitiones Arithmeticae* have been translated into German (Mauser, Berlin, 1889) and into French, and should be taken up early by the student of this subject.

Mention may also be made of:

Bachmann. Zahlentheorie. Vol. I., Leipzig, 1892.
Cahen. Theorie des Nombres. Paris, 1900.

Elementary Foreign Texts.

The German works named are all of university grade, and the French works are usually intended for corresponding work, as given in the most advanced courses in mathematics in the *lycées*. Occasionally an American teacher is interested in seeing a specimen of the texts used in the intermediate and lower work in Germany and France. As such may be mentioned:

In German.

Harms und Kallius. Rechenbuch. Twentieth Edition, Leipzig, 1899.

Mehler. Elemente der Mathematik. Twentieth Edition, Berlin, 1896. (Very popular and widely used. It covers the entire ground of secondary mathematics, giving theory only.)

Bussler. Elemente der Mathematik. Dresden, 1897. Covers the entire ground; the matter is divided into sections containing the topic to be taken up in each year of the gymnasium. There is a companion book of exercises.

Kambly. Algebra, Plane Geometry, Trigonometry, Solid Geometry. Breslau. A very popular series — new editions constantly appearing. The geometry has reached more than a hundred editions.

Bardey. Algebraische Aufgabensammlung. Leipzig. A very excellent and much used collection of good algebraic problems, which reached its twenty-third edition in 1897. In 1900 a parallel edition, bringing data into conformity with modern life and paying more attention to applications to geometry and the natural sciences, was issued by Pietzker and Presler.

In French.

Leysenne. Arithmétique. Tenth Edition, Paris, 1901.

Bourlet. Arithmétique. Third Edition, Paris, 1903.

——— Algèbre. Paris, 1903.

Borel. Arithmétique, Algèbre. Paris, 1903.

Hue and Vagnier. Géométrie. Sixth Edition, Paris (no date).

——— An abridged edition for girls' schools. Second Edition, Paris, 1898.

General Addresses.

Addresses and papers by leading mathematicians and other scientists and educators treating their subject in a general way are always stimulating and instructive, and the teacher should read all on which he can lay his hands. In the mathematical papers, technical mathematics will often be found which may be unintelligible to the reader, but there will be enough that can be well understood to repay fully the requisite expenditure of time, thought, and perhaps money.

As instances of papers of this character the following may be named:

Bôcher. The Fundamental Conceptions and Methods of Mathematics. Bull. Am. Math. Soc., 1904, pp. 115–135.

Carus. Mathematics, the Old and the New. Open Court, **2**: 1468.

Farrar, F. W. General Aims of the Teacher. Cambridge, 1883.

Henrici. Presidential Address. 1883. (Reports of British Association.)

Hilbert. Mathematical Problems. Address at International Congress of Mathematicians, Paris, 1900. Translated by Mary W. Newson, Bull. Am. Math. Soc., 1902, pp. 437–479.

Klein. Evanston Colloquium, 1893.

Klein. Present State of Mathematics. Monist, 4.

Moore. Presidential Address, 1903. (References in Chapter VI.)

Newcomb, S. Mathematical Thought. Nature, **49**: pp. 325–329.

Perry. Glasgow Address, 1901. (Reference in Chapter VI.)

Picard. On the Development of Mathematical Analysis, and its Relation to Certain other Sciences. Translated by M. W. Haskell, Bull. Am. Math. Soc., 1905, pp. 404–426.

One of the chief benefits derived from the teacher's personal reading is that it keeps alive his heuristic spirit, provided the work is not merely acquisitive, but also creative, that is, if the reading consists merely of understanding what is presented in the book read, but also includes working out problems, theorems, and applications independently. These may be suggested by the book itself or they may be thought of by the reader. In the latter case it is possible that he may find something which is really new, and may thus enlarge the domain of knowledge. This is, of course, the most pleasant culmination, but one must be prepared to find that the discovery — one's own *bona fide* discovery — had been anticipated by another, perhaps long before, and may even be quite well known. Even the specialist has this experience many a time, and the teacher who is only on the threshold of the mathematical domain will rarely avoid it; but this does not detract from the intrinsic value of the discovery or the exhilaration of making it, and should not deter the teacher from engaging in such work, any more than analogous considerations prevent him from insisting that his pupils work out many things for themselves which are common property.

The Reader's Own Work.

The objection may be raised that the teacher of mathematics who is acquainted with the higher regions of the subject will be likely to shoot above the heads of his pupils. That there is danger of this cannot be denied, but the safeguard against it is not ignorance on the part of the teacher, but a careful study of the child mind and of the particular minds before him, great caution to avoid matter beyond the grasp of the pupils, and constant tests of various sorts to determine whether or not the pupils have grasped the matter in hand. The best safeguard is to teach in the heuristic spirit, and to shun all occasion, however tempting, to drop into the lecture mode.

An Objection considered.

What has preceded relates to the scientific preparation of the teacher. The actual teaching of mathematics is an art, — requiring knowledge of the science, but none the less an art. The secondary school teacher has been less fortunate than his colleague of the earlier school years, for whose training in the art of teaching many good normal schools exist and have existed for a long while; but even though the ranks of secondary teachers are recruited largely from the students and graduates of colleges and universities, the last decade only has witnessed a beginning on the part of these institutions to provide courses of instruction in the pedagogy of secondary mathematics and the founding of two institutions — The Teachers' College, Columbia University, and The School of Education, the University of Chicago — bearing somewhat the same relation to the preparation of secondary teachers that the normal schools do to the preparation of grade teachers.

Pedagogic Preparation.

Perhaps one reason for the fact that not more opportunities have been provided for the training of the secondary teacher lay in the well-grounded fear that purely theoretic pedagogic courses, such as must in general be given in colleges and universities, are too much hampered by the lack of the practical element. This is very true so far as it relates to inexperienced beginners, but is much less serious when, as is often the case, the students have perforce put the cart before the horse, and come to the study of the pedagogic side of mathematics with considerable practical experience behind them. In time this abnormal state of affairs will be gradually outgrown as more and more secondary teachers enter upon their work with sufficient pedagogic training. It is obvious that such training is needed,[5] and with the growth of the facilities for obtaining it the number of pedagogically untrained beginners admitted to the ranks of secondary teachers should correspondingly decrease.

Why Facilities have been Lacking.

No one questions that in the pedagogic training of teachers the theoretic study of pedagogy and the practice of the art of teaching should go hand in hand. The present tendency in America is to associate a secondary school for observation and practice

Theory and Practice Hand in Hand.

[5] See Chapter I.

with the university, as, for example, in the two institutions named above.

What can be done in the Secondary School.

What can be done in connection with the secondary school or the college depends on local conditions. Many colleges have preparatory schools in direct connection, in which secondary teaching could be practically studied. In the larger secondary schools it would seem at least possible to arrange some sort of friendly and informal supervision of the work of each beginner by one of his more experienced colleagues. This should include frequent interchange of class visits and numerous personal conferences on what has taken place in these class sessions, plans for subsequent sessions, discussion of more general questions of teaching and of subject matter. While informal, the work should be taken seriously, and the supervising teacher should, if possible, be relieved of a corresponding amount of other work.

Young teachers would as a rule meet a proffer of such assistance more than half way, and in general it may be said that all the facilities which have been offered in this country for the study of pedagogy of secondary mathematics have been eagerly seized. If none of them are accessible to any particular teacher, active or prospective, he can at least take up some line of reading by himself, and if inexperienced and not yet at work he can profit much by frequent visits of observation to all the mathematical classes within his reach. He would be cordially welcomed everywhere, without doubt, and any questions that might suggest themselves to him concerning the mode of instruction would be gladly answered by the teachers privately. He would of course remember that such replies are personal favors at the expense of the teacher's scant leisure time, and would be careful not to become burdensome.

What has been done in Prussia.

Beyond doubt, by far the best work that has hitherto been done in the way of training secondary teachers is that of Prussia, extending over two years and carried on in connection with secondary schools, in which the candidate lives for two years, with daily opportunity to observe the actual work and, when prepared, himself to essay teaching under guidance.[6]

Even with this extended preparation the leaders of the present movement for the improvement of the teaching of mathematics in Germany urge better preparation of teachers as one of the fundamental improvements.[7]

Daily Preparation. No general preparation, however thorough, can dispense the teacher from careful preparation day by day for the work of each class period. A general, even though accurate and thorough, knowledge of the subject matter is not sufficient. Special preparation for each class hour is requisite. This would include:

1. General plan of the whole course at the outset.
2. Plan in more detail of the work of the period next ensuing, say the next week or two.
3. Refreshing of all the minutiæ of the subject matter to be taken up in the class period.
4. Plan of the mode of treatment. This includes the mode of meeting needs of particular pupils, as well as the handling of the class as a whole.
5. Determination of the assignment for the next time.

Though it will be possible as a rule to make the assignment as planned, the prime requisite is that the topic in hand shall have been sufficiently grasped by the class on the whole. It is folly to go on unless this is the case, despite the fact that the teacher's plans or even the curriculum may call for it.

Mathematical Clubs

Organizations of Teachers and Pupils. Special organizations holding stated meetings for the consideration of mathematical interests in a broader way than the class work permits may prove very valuable adjuncts of the instruction. In the sequel such organizations are called "mathematical clubs."

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[6] For details, see Young, *Teaching of Mathematics in Prussia*, pp. 18–22.

[7] For an outline of this movement as exhibited in some recent publications, see Young, *Bull. Am. Math. Soc.*, 1906.

They may be intended primarily for the pupils or for the teachers. Clubs of both sorts have been conducted with success in secondary schools.

The clubs for pupils of which I have personal knowledge have had their membership restricted to pupils of the last school year. The sessions are held in the evening, usually fortnightly, and membership is of course voluntary. A large percentage of those eligible for membership joined the club and attended the meetings regularly. At the sessions, papers and reports on assigned topics were presented by the pupils belonging to the club, and discussed. In a club in a large

Topics for Teachers.

high school the papers by the pupils were supplemented by addresses by the teachers of various subjects, showing the bearing of mathematics on their subjects. When this is done, the selection of subjects (within the general scope of the club's field of work) is of course made by the teachers who may be secured to give the lectures. In the instance mentioned some of the subjects were:

Mathematical theory of engineering instruments.
General principles of railroad engineering.
The social and engineering aspects of the location of railroads in South America, Africa and Siberia.
Analytic geometry historically considered.
Methods in the calculus.
History and philosophy of the calculus.
The theory of equations.
Mathematical electricity.
Ready rules in mensuration.
Mathematical games.
Mathematical theory of the chemical balance.
The quantitative in science.
Mechanics of organic motion, — plants, animals.
The Peaucellier and allied linkages.
Dynamics of geology.
Mathematical astronomy.
Mathematical concepts and their validity.

The topics assigned to the pupils must be quite specific, with detailed directions as to what is expected. The work done will usually consist of reading what is said on the topic in various works on the history of mathematics which may be accessible; the facts so ascertained will constitute the basis of the report. They will be woven together into a more or less coherent whole according to the pupil's tastes and abilities. It is probably better to give definite page references and not simply say, "Look up what Ball says."

Topics for Pupils.

Historical topics. ("The history and development of," or some analogous expression is to be understood as prefixed to each topic.)

Arithmetic.

The number symbols.
Measures of time; of angles; the decimal system.
The four fundamental operations with integers.
Prime numbers.
Properties of integers.
Fractions.
Applications of arithmetic.

Algebra.

The name "algebra."
The symbols of algebra (and arithmetic).
The use of letters to represent numbers.
Zero and infinity.
Negative numbers.
Irrational numbers.
Imaginary numbers.
Involution.
Evolution.
Unknown quantities; equations.
Equations of the second degree.
The binomial theorem.

Geometry.

The terms of geometry.
Parallels.
Triangles.
Constructions with rules and compasses.
The circle.
The computation of areas.
π.
The trisection of an angle.
The duplication of the cube.

Trigonometry.

The six trigonometric functions.
The names and symbols for the six functions.
The fundamental formulæ.
The discovery of logarithms.
Tables.

The historical subjects named above call for the presentation of what can be ascertained as to the development of the mathematical topics named irrespective of time or nation. Topics may also be assigned calling for a report on the work of a nation or of a period. For example, mathematics among the Chinese, the Hindoos, the Egyptians, the Romans, the Arabs, might each constitute a topic. Early Greek mathematics, the Pythagorean school, the Alexandrian school, are examples of analogous topics. The divisions of several of the books on the history of mathematics are on these lines and will suggest other topics.

Topics of Subject Matter.

Assignments may also call for a fuller discussion of some topic or method than is desirable in a class, or the collation and presentation of various proofs of the same thing; for example:

Various proofs of the Pythagorean theorem.[8]
The solution of equations of the first degree.
Various solutions of equations of the second degree.[9]
The solution of the cubic equation.
The solution of the biquadratic equation.

The last two topics involve the presentation of new matter, and should be confined to a brief presentation and discussion of a standard solution as found in current books. So limited,

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[8] For material see *American Mathematical Monthly;* Edwards' *Geometry;* Rupert, *Famous Geometrical Theorems and Problems*, Boston, 1900.

[9] An abundance of such solutions are contained in Matthiessen, *Grundzüge der antiken u. modernen Algebra*, Leipzig, 1878. This is a volume of over a thousand pages, devoted exclusively to various solutions and methods of solution of equations of the first four degrees.

these topics are quite within the grasp of such secondary school pupils as would wish to participate in a mathematical club. These topics would be of interest as widening the range of the equations solved and also as completing the list of the general equations that are solvable algebraically. One or two geometric problems leading to cubic equations might also be assigned.

More special topics and particular problems may also be taken up if the class exercises have awakened interest in them, or if they connect in an interesting way with the class work. Plotting a few curves will be interesting and also an instructive review of various properties of polynomials and equations. Plotting the polynomials of the second degree is finding favor with secondary teachers and text-book writers, and no additional knowledge or skill is needed to plot expressions like —

$$y = \frac{1}{x}$$
$$y = \frac{x}{1 + x^2}$$
$$y = 2x^2 - x^3$$
$$y^2 = x^3$$
$$y^2 = x^3 + x^2$$
$$y^2 = x^3 - 3x^2 + 2x$$
$$y^2 = x^3 - x^2,$$

while the striking shapes of the resultant curves and the connection between them and their equations always interests pupils.[10]

Models and Instruments.

The exhibition and explanation of models and instruments constitutes another interesting class of possible topics. If theodolite, slide rule, sextant, etc., are available, a member of the club could make a study of the theory and uses of one of these and report. Various linkages are easily made from cardboard, and have interesting mechani-

[10] Other material that might be used is given in Moore, Cross-section paper, etc., *Sch. Rev.* 1906.
See also Chap. VI.

cal applications, for example, to transform rectilinear into rotary motion.

Sketches of the life and work of mathematicians, ancient and modern, whose work has had a bearing on the secondary field, also offer opportunities for work. The life of Plato, Euclid, Archimedes, Apollonius, Hero, Diophantus, Gerbert, Leonardo of Pisa, Jordanus Nemorarius, Stifel, Descartes, Vieta, Leibnitz, Newton and Euler may be named as instances.

The works on mathematical recreations cited above open a wide field of topics and furnish much material relative to curious properties of numbers, mathematical games, problems relative to these and to other known games, as checkers, card tricks, curious geometric problems, paradoxes, etc. Interest may be lent to the club sessions by permitting the members to look up and present some card trick or the like dependent on mathematical principles; the performer may mystify his audience for a while, but if no one in the audience sees through the trick and explains it, the performer must finally do so.

Social features. I do not know of any instance in which social features were added to the sessions, but through not doing so a legitimately attractive feature was certainly lost. Similar organizations among the students of German universities — who are much older and have much more serious scientific interest than the pupils in secondary schools — are largely held together by their social features. Even societies of professional mathematicians do not neglect the social side at their various meetings, and there is no apparent reason why the meetings of immature pupils should outdo those of their elders in the severity of their devotion to the purely scientific and technical. Teachers would find in the informal parts of the meeting excellent opportunity for closer personal relations with the pupils.

A club for the teachers, which is only possible, of course, where there are several teachers of mathematics in the same school, might serve several useful functions; it should promote close personal acquaintance and friendship among the teachers, and might also be a sort of a faculty meeting where questions relative to mathematical instruction in the school would be considered, and also more general topics taken up. The papers may relate to problems of the pedagogy of mathematics or to mathematical subject matter. They may be formal or informal, though there should always be careful preparation. The joint study at the sessions of some subject of more advanced mathematics may be very valuable (see list earlier in this chapter).

Clubs for Teachers.

All these matters would depend on the preferences of the teachers and other local conditions. It is not necessary that the organization and programmes be formal and elaborate, — it is probably better that they should not be so; but however organized and conducted, it would seem possible for a large part of institutions with several teachers of mathematics to have some sort of mathematical club among the teachers, which would be of signal benefit to the institution as well as to the teachers personally.

CHAPTER IX

THE MATERIAL EQUIPMENT

The Library

Books for Teachers and for Pupils.

THE selection of books for the library must have in view the needs of the pupils and those of the teachers. The complete separation of the books intended for the use of the pupils from those of the teachers, making two quite distinct libraries, as is customary in the secondary schools of Germany and of France, is advantageous when, as there, the institution is able to set apart working and social rooms for the exclusive use of the teachers, of which at least one can be used as a library. Equipped with suitable tables, this room would be an excellent workshop in which the teachers could quietly use books without any formality. Unless these very desirable facilities can be supplied, there would seem to be no need to separate the two classes of books. In mathematics especially such a separation would cause considerable duplication. Most of the use made by pupils of mathematical books in the library must somehow or other be stimulated by the teacher, usually by specific reference, and in consequence the pupils will to a large extent use the same books as the teachers.

What books to select. For the pupils, a collection of the best current texts is desirable, and it should be kept up to date. Pupils should be encouraged to make free collateral reference to these books, and to bring up points of interest in the class. Definite assignments for such reading and report may sometimes be made. It would not be amiss to include one or two of the best German and French texts, both for teachers and such pupils as would be able to use them. There are likely to be in almost every class some pupils who either bring from home or have acquired in the school the ability to make a little use of these books. Such use, or even the reference, serves to widen the pupil's views both from the point of view of mathematics and of modern languages (see the lists below and in the preceding chapter).

The library should also include books on the teaching and study of mathematics, on mathematical recreations and on miscellaneous subjects relating to mathematics. The bibliographies which have preceded will suggest possibilities for the library. Some of the following works may also be useful. The teacher will, of course, select those of which he can best make use.

History.

Cantor. Vorlesungen üb. Geschichte d. Math. Second Edition, 3 Vols., Leipzig, 1894–1900.

This is the most important single work on the history of mathematics. Other writers have drawn on it largely.

Gow. A Short History of Greek Mathematics. Cambridge, 1884.

Marie. Histoire des Sciences mathématiques et physiques. 12 Vols., Paris, 1883–1888.

Allman. Greek Geometry from Thales to Euclid. Dublin, 1889.

Cajori. The Teaching and History of Mathematics in the United States. Washington, 1890.

Boyer. Histoire des Mathématiques. Paris, 1900. With a number of portraits of mathematicians.

Hoefer. Histoire des Mathématiques. Fifth Edition, Paris, 1902 (pp. 602).

Chronologically arranged. Covers advanced as well as elementary mathematics.

Tropfke. Geschichte der Elementar-Mathematik in Systematischer Darstellung. Leipzig, 1903, 2 Vols, containing together 844 large pages.

This is by far the most satisfactory and thorough work on the history of elementary mathematics that has yet appeared. It covers the ground from the elements of arithmetic to analytic geometry, inclusive. The matter is arranged according to topics, not chronologically.

Fink. A Brief History of Mathematics. Translated by Beman and Smith. Chicago, 1900.

Conant. The Number Concept, its Origin and Development. New York, 1896.

On development of the number concept, words, symbols and systems of notation among primitive peoples.

Ball. Short Account of the History of Mathematics. Third Edition, London, 1901 (pp. 527).

Ball. Primer of the History of Mathematics. London, 1895 (pp. 146).

Cajori. History of Mathematics. New York, 1893.

Cajori. History of Elementary Mathematics. New York, 1896.

Heath. Diophantos of Alexandria. Cambridge, 1885.

Pedagogic, Philosophical, General.

De Morgan. The Study and Difficulties of Mathematics. First issued 1831; reprinted, Chicago, 1898.

LaGrange. Lectures on Elementary Mathematics. Translated by McCormack. Chicago, 1898.

Klein. Lectures on Mathematics, delivered in connection with the World's Fair at Chicago. New York, 1894.

Clifford. Common Sense of the Exact Sciences. New York, 1885.

De Morgan. Elementary Illustrations of the Differential and Integral Calculus. First published 1832; reprinted, Chicago, 1899.

Richard. Sur la Philosophie des Mathématiques. Paris, 1901.

Discusses rather simply some of the more philosophical notions which are nowadays receiving much attention. Axioms, Postulate of Euclid, Infinity, Continuity, Matter, Dimensions of Space, etc.

Du Bois Reymond. Die Allgemeine Functionen Theorie. Metaphysik; Theorie der Mathematischen Grundbegriffe; Grösse, Grenze, Argument and Function. Tubingen, 1882.

Metaphysical and abstract, but not involving so much technical mathematics as other works of this character.

Dedekind. Essays on the Theory of Numbers. I. Continuity and Irrational Numbers; II. The Nature and Meaning of Numbers. Translated by Beman. Chicago, 1901.

Fine. Number System of Algebra. Boston, 1890.

Stallo. The Concepts and Theories of Modern Physics. New York, 1884.

A considerable part of the work is devoted to the relations between the concepts of physics and of geometry.

Geometry.

Euclid. A good edition of Euclid is desirable. Various good editions are published in England.

Mention may also be made of the German edition of **Simon** (Euclid u. die sechs planimetrischen Bücher. Leipzig, 1901), which is valuable on account of its annotations.

Dodgson. Euclid and his Modern Rivals. London, 1885.

Henrici u. Treutlein. Lehrbuch der Elementargeometrie. Leipzig, Second Edition, 1891.

Junghans. Lehrbuch der ebenen Geometrie. Berlin, 1879.

Loria, G. Della varia fortuna di Euclidi. Rome, 1893.

Ingrami. Elementi della Geometria per le Scuole secondarie. Bologna, 1899.

Enriquez e Amaldi. Elementi di Geometria. Bologna, Second Edition, 1905.

Guichard. Traité de Géométrie. Paris, Second Edition, 1903. Two large volumes (883 pages together).

A fair sample of the French treatises. The first volume contains ordinary plane and solid geometry, the second the "Compléments," including much modern geometry, the whole conforming to the new curricula of 1902, and showing the extent to which the study of geometry is carried in the French *Lycées*.

Fontenée. Géometrie dirigée. Paris, 1897.

Discusses use of signed magnitudes in geometry, line segments, angles, volumes; intended for pupils.

Row. Geometric Exercises in Paper Folding. Edited by Beman and Smith. Chicago, 1901.

Collections of Problems.

Meier Hirsch. Aufgaben Sammlung. Twentieth Edition. Altenburg, 1890. A collection of problems in algebra.

The first edition appeared in 1804, and may be regarded as the model of subsequent collections.

Bardey. Algebraische Gleichungen nebst den Resultaten u. den Methoden zu ihrer Auflösung. Fourth Edition, Leipzig, 1893.

A work of 390 pages, giving more difficult equations solvable by quadratics with results, and in many cases with full solution and discussion. There are 492 problems with one unknown, 394 with two unknowns, 114 with three and more unknowns. The problems are generally too difficult for class use, but will be interesting to the teacher and may serve as supplementary material for bright pupils.

Laisant. Recueil de Problèmes de Mathématiques. Paris, 1893.

A collection of problems which have been proposed and solved in various mathematical journals since 1842, with references. In seven parts. The first part, which can be obtained separately, covers arithmetic (only a few problems), elementary algebra and trigonometry.

Dodgson, C. L. (Author of "Alice in Wonderland.") Pillow Problems thought out during Wakeful Hours. London, Fourth Edition, 1895.

Seventy-two problems with solutions. All but five from algebra, geometry and trigonometry.

Wolstenholme. Mathematical Problems. London, Third Edition, 1891.

Contains 2,814 problems, of which perhaps two to three hundred fall into the secondary field; the others are of collegiate grade.

Milne. Weekly Problems Papers. London, 1891.

Of collegiate grade.

Mathematical Recreations.

Ahrens. Mathematische Unterhaltungen und Spiele. Leipzig, 1900.

Bachet de Meziriac. Problèmes plaisants et delectables qui se font par les nombres. Paris, 1884.

A reprint, reviewed, simplified and augmented, of an old work first published in 1612.

Ball. Mathematical Recreations and Problems. Third Edition, London, 1890.

Cavendish. Recreations with Magic Squares. London, 1894.

Fourrey. Récréations Mathématiques. Paris, 1899.

Grosse. Unterhaltende Probleme u. Spiele in Mathematischer Beleuchtung. Leipzig, 1897.

Hatton. Recreations in Mathematics. Second Edition, London, 1840.

Latoon. On Common and Perfect Magic Squares. Cambridge, 1896.

Lucas. Récréations mathématiques. 4 Vols., Paris, 1891–96.

Lucas. L'Arithmétique amusante. Paris, 1895.

Mittenzwey. Mathematische Kurzweil. Third Edition, Leipzig, 1895.

De Morgan. A Budget of Paradoxes. London, 1892.

Schubert. Mathematical Essays and Recreations. Translated by McCormack. Chicago, 1898.

Vinot. Récreations mathématiques. Fourth Edition, Paris, 1898.

Viola. Math. Sophismen. Second Edition, Wien, 1886.

Flatland. Published in Boston anonymously.

Perry. Spinning Tops. London, 1901.

Quotations.

Ahrens. Scherz u. Ernst. i. d. Mathematik. Leipzig, 1904.

Maupin. Opinions et Curiosités touchant la Mathématique. Paris, 2 vols., 1898–1905.

Rebière. Mathématiques et Mathématiciens, Pensées et Curiosités. Second Edition, Paris, 1893.

Journals.

School Science and Mathematics. Chicago.

The organ of various American associations of teachers of mathematics. It should be in every school library. Papers discuss pedagogic questions rather than subject matter of mathematics.

American Mathematical Monthly. Springfield, Missouri.

Papers on topics from mathematical subject matter, many of which are within the range of the secondary teachers' interest. Proposal and solution of problems a prominent feature.

The Mathematical Gazette. London.

The organ of the Mathematical Association.

Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht. Leipzig.

L'Enseignement mathématique. Paris.

An international journal for the teaching of mathematics.

Annals of Mathematics. Cambridge, Mass.

Publishes papers on mathematical topics; more advanced.

The secondary school library should subscribe to as many of the above as circumstances will permit.

L'Éducation mathématique, Le Journal de Mathématiques élémentaires, and Le Revue de Mathématiques spéciales.

These three journals, all published at Paris, are devoted to proposal and solution of problems, and the publication of the papers set in mathematics in the various French schools. The first covers the range of topics of American secondary schools (arithmetic, algebra, elementary geometry), the other two are respectively of elementary and advanced collegiate grade.

Of the journals devoting a portion of their space to secondary mathematics and its pedagogy may be mentioned:

Jahresbericht der deutschen Mathematiker-Vereingung, Leipzig.
Unterrichtsblätter für Mathematik u. Naturwiss., Berlin.
Zeitschrift für Realschulwesen, Wien.
Archiv für Mathematik u. Physik, Leipzig.

In closing this list, attention is again called to its character as supplementing the references given in the separate bibliographies and elsewhere. Though the list contains some works that have been mentioned elsewhere, its main purpose is to name and classify roughly, for library purposes, some works that have not been otherwise cited.

The Teacher's Needs.

While under current American conditions there would be little advantage in installing the teacher's library apart from that of the pupils, the teacher's needs should none the less receive separate attention. No doubt the teacher, even more than the pupils, will use the books intended also for the latter, but a large part of the expenditure may well be for books entirely beyond the reach of the pupils. No suggestions can be made in a general way. The work of each individual teacher will determine what is best for his school library. Whatever books he actually uses are suitable; none should be bought of which he does not make some use. The chapter on the preparation of teachers suggests lines of reading and related books. If the teacher is able at times to take up residence in some university, or to work under guidance in university extension courses or otherwise, his instructors will give him useful references and suggestions. Where there are several teachers in a school, the needs of even one will be sufficient for the inclusion of a book, though energy as well as money will be economized by making common cause (see previous chapter).

The Teacher's Personal Library.

No special remarks need be made as to the teacher's personal library. It will be stocked in accordance with his tastes and the length of his purse. Most teachers will desire to own the books with which they work most, so as to have them always at hand in the workshop, and to be free to annotate them. Useless duplication between the school library and the teacher's library will of course be avoided.

The other Material Equipment

The Mathematical Laboratory and the Mathematical Museum.

The equipment of the mathematical laboratory has already been discussed. A museum will grow up naturally in connection with the class work, provided only that the museum *exists*, even if it is merely an empty shelf at first, and that teachers and pupils are reasonably on the alert for opportunities to enrich it. It may include models, some purchased, others made by pupils, drawings of great variety, busts and portraits of mathematicians, linkages, apparatus to illustrate various mathematical concepts and mathematical machines of various sorts.[1]

The Class-room Equipment.

It would seem superfluous to say a word about the rudimentary class-room equipment were it not that so many rooms are far from well equipped. It may therefore be permitted to mention that the mathematical class-room should be supplied with seats arranged so that when seated the pupils receive the light from the left, and so that any pupil can pass to and from the board without disturbing

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[1] Some portraits of mathematicians are published by the Open Court Co., Chicago. A list of elementary mathematical apparatus is given in *Zeitschrift für Math. u. Naturw. Unterricht*, 1903, p. 67. Lists of arithmetical apparatus are given in Kehr, *Praxis der Volksschule*, 1880, p. 209; Unger, *Arithmetik*, p. 203.

any other; all furniture fastened immovably to the floor; all seats so rigid as not to *squeak* no matter how the pupil may *squirm;* floor covered with linoleum to reduce noise of walking to minimum; ample blackboard for all to work simultaneously, of such texture and so lighted as not to reflect light and appear "shiny" from any point of view; some squared board; an eraser for each pupil; pointers; rulers (those for board use having handles on the back); wooden protractors and chalk compasses; slated globes; colored chalk (serviceable colors, as red, orange, yellow, light green, not assorted boxes); cardboard and thread for making models.

CHAPTER X

THE CURRICULUM IN MATHEMATICS

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What Mathematics should be studied?

IN the chapter on the purpose and value of the study of mathematics we have considered the question "Why should mathematics be studied?" Here we come to the question "What mathematics should be studied?" Evidently the subject matter must be so selected that the ends of the study of mathematics can be attained, but fortunately this still permits a wide range of choice.

The selection should be such as:

1. To exhibit most clearly and to best advantage the mathematical type of thought.

2. To help to a better understanding of the laws of nature.

3. To bring out distinctly the mathematical relationships that exist in the social organism and in the activities of modern life, and to show how mathematics aids in solving their problems.

4. To give sufficient skill in the actual performance of mathematical processes to meet the future needs of the pupil.

5. To permit the organization of the material into a homogeneous whole, meeting the demands of scientific pedagogy.

The customary subjects arithmetic, algebra, geometry, trigonometry conform in a general way to these desiderata. In the selection of particular topics, the first head justifies and requires the inclusion of such as exemplify clearly the nature of mathematical reasoning, and the sequence of its results as a logical system. But whenever a topic is so complicated that the pupil is likely to lose the chain of reasoning in the mechanism, the first head gives no warrant for its inclusion in the course. Unless one of the other heads demands it, which is seldom the case, it may well be omitted altogether.

Close Connection with Modern Life.

The second and the third head call for the inclusion of topics and methods, illustrations and problems relative to the world of men and of matter, as it can be seen and understood by the pupil. The trend in the making or modifying curricula throughout the world to-day is to lay more stress on the inclusion of such topics. This need not mean that less value is attached to the strict logic of mathematics, or that it is made less clear and prominent to the pupils. On the contrary, it may well be claimed that by the diminution of merely technical manipulation, by omitting problems whose sole merit is their complexity, by arousing the pupil's interest through the connections of abstract mathematics with concrete environment, the logical aspects of the subject are made to stand out all the more clearly.

The questions as to curriculum relate not only to what is to be taught in each subject, but also to the order in which the subjects themselves are to be taught. No sharp line can be drawn between these categories of questions, and some of their aspects may be considered to better advantage in connection with the teaching of the various subjects themselves. Reserving questions of detail for these later chapters, only general questions need be taken up here.

In the Grades. In the grades various proposals for modifications of the current curriculum have been made; such as the elimination of outgrown and less important matter from arithmetic, the bringing of the latter into conformity with the needs of present-day life, and the introduction, as phases of arithmetic, of the equation, the elements of literal arithmetic, the numerical side of geometry, constructive and inventional geometry, and perhaps the elements of demonstrative geometry. The detailed consideration of these questions may be best deferred to the chapter on the teaching of arithmetic.

In the High School. Turning to the curriculum in mathematics in the secondary school, we mention first a list of topics in the various subjects representing the standards of the best secondary instruction of the day given in the report of a committee of the American Mathematical Society, 1903. This list will be discussed in connection with the various subjects.

Tendencies: In England. It is profitable to note also the changes in curricula in leading European countries. The last decade has witnessed remarkable changes in the official curricula of England, France and Germany. In England the change has markedly affected geometry. The situation is practically controlled by the examination requirements of the Universities, and these have been modified so as to permit the replacement of Euclid as a text by modern works written with the boy's capabilities in mind, and combining the concrete with the logical so as to appeal more strongly to the pupil's interest. The instruction in geometry has been generally modified along these lines throughout England.

In France. In France the new curricula extend the minimum of mathematics required of all students in the *lycées* from the elements of trigonometry to the elements of the calculus, with stress on graphic methods and practical applications. The character of the last year's work may be seen by examining Tannery's *Notions de Mathématiques*, a work well worth study.[1]

The current of thought which has influenced recent French changes is well sketched by this writer:

"One has not even a slight idea of what mathematics is, one does not suspect its extraordinary scope, the nature of the problems that it proposes and solves, until one knows what a function is, how a given function is studied, how its variations are followed, how it is represented by a curve, how algebra and geometry aid each other mutually, how number and space illustrate one another, how tangents, areas, volumes are determined, how we are led to create new functions, new curves, and to study their properties. Precisely these notions and methods are needed to read technical books in which mathematics is applied. They are indispensable to whoever wishes to understand the rapid scientific movement, the manifold scientific applications of our times which day by day tend to modify more profoundly our fashion of thinking and of living.

"They are simple and easy so far as essentials are concerned, easier than many demonstrations that we do not hesitate to give to pupils, demonstrations that are long and complicated and that have no bearing beyond what they prove. These methods should penetrate more and more into elementary instruction, both to abridge and to strengthen it. No doubt we must think of the development of the mind, but does any one think that limited and particular methods, that questions which the pupils vaguely feel are useless or factitious, will contribute more to it than general methods? And if the pupils see the power of these methods, will they be less disposed to take some pains to master them and their applications?"[2]

In Germany.

In Germany, the new curricula of 1901, while making no change of any consequence in the subject matter taken up, direct the omission of elaborate computations which find little subsequent application; for example, division of one long polynomial by another, and lay

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[1] Tannery, J., *Notions de Mathématiques*, Paris, 1903.

[2] Preface, p. vi.

more stress on concrete (propædeutic) methods, geometric drawing, geometric constructions, the function-concept and the applications of mathematics in the other sciences and in practical life. In 1905, there was made public a report of a committee of the German Society of Natural Scientists and Physicians. It is a document of the first importance. In the mathematical section, indeed in the constitution and work of the committee as a whole, F. Klein, professor of mathematics at the University of Gottingen, was a leading spirit. This report outlines a mathematical curriculum emphasizing still more the tendencies which were marked in the curricula of 1901. It recommends still further elimination of abstract and merely technical matter, the introduction of each subject in a concrete way, an earlier and progressive use of graphic methods, models and instruments (*e. g.*, slide rule), attention throughout also to the practical and numerical aspects, applications to physics and mechanics, and (optionally) the introduction of the elements of the calculus.

This report has been too recently published to have as yet affected the curriculum in actual use, but that it will have decided influence in shaping the curricula of the near future cannot be doubted.

In America.

In America, improvements of the curriculum along the same general lines as those followed in the European countries named are also possible;[8] as a matter of course, they would be introduced and carried out, not in mere imitation of what others have done, but in accordance with our own needs and in harmony with the spirit of our institutions. We have in addition a conspicuous and easy reform to make that these other countries have long since accomplished,

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[8] A good rapid survey of the curricula in American secondary schools can be found in Brit. Special Reports, XI, part 2, 1902.

(a) Sanford (Brookline, Mass.), *Curriculum of American Secondary School*, pp. 1–22.

(b) Hanus, P. N., *Secondary Education in a Democratic Community*, pp. 23–66. (Tables in his Appendix.)

namely, simultaneous instruction in algebra and geometry. The general custom of devoting the first year to elementary algebra exclusively, the second to plane geometry exclusively, the third (or fourth) to solid geometry and the continuation of algebra, and the fourth (or third) to physics, has but little to commend it. Algebra is more abstract than geometry, it has fewer points of contact with real life, its reasoning is more difficult than that of the easier demonstrations of geometry.[4] Further, the more difficult parts of either subject are harder than the easier parts of the other, and each can be made of valuable help in the development of the other. The ideas of both are assimilated slowly by the mind, requiring not so much daily use as occasional use in various ways (quite easy for some time) throughout a long period. To rush a pupil through a subject at high pressure with the exclusion of all other mathematics may train the pupil to pass examinations, but it will not develop the largest measure of thought power or be conducive to real assimilation of the subject.[5]

Simultaneous Teaching of Algebra and Geometry.

The conclusion follows naturally that algebra and geometry should be taught side by side; that if under pressure of circumstances either subject must be taken up first, it should be geometry. For many years the tandem order has been under

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[4] "The elements of plane geometry should precede algebra for every reason known to sound educational theory. It is more fundamental, it is more concrete and it deals with things and their relations rather than with symbols." — Butler, N. M., *Meaning of Education*, New York, 1898, p. 171.

[5] "That algebra and arithmetic be taught parallel with geometry is not merely desirable, but is indispensable to preserve in the totality of the mathematical work that character of unity and coordination without which it loses all interest and all worth. A youth who learns first arithmetic, then algebra, and then geometry, and who keeps on in this way for ten years, will have his mental powers less strengthened than by three or four years of parallel instruction, intelligently conducted." — Laisant, *La Mathématique*, p. 227.

question,[6] and the weighty objections to it and the strong reasons in favor of another order have repeatedly been urged, though with little general effect. There have been isolated private schools which have ventured on a better order,[7] but they remain exceptional.

The reasons for the failure to make the improvement seem to be those of inertia, of conservatism, rather than a conviction that the change proposed is not good or that the customary order is better. I know of no published defence of the traditional procedure, but still the old order, which finds no defenders in theory, persists in practice.

Writers, committees, associations, have recommended change. The time to act has come. Fortunately this is a change

What to do.

which can be effected in any single school without disturbing its relations with either the colleges and technical schools for which it prepares or the grade schools which furnish its pupils. The time is ripe for single schools, acting independently, to rearrange their own curriculum in mathematics. The change can be made within the mathematical work alone without disturbing the rest of the

Action by Single Schools possible.

curriculum. It consists merely in beginning both algebra and geometry at the outset and carrying them side by side through the first two years. The third (or fourth) year already contains algebra and geometry (solid), so that the change here would be even less,—

[6] See Cajori's *Teaching of Mathematics in the United States*, pp. 352–359.

[7] In Phillips Andover Academy, for example, the order is

	IV.	III.	II.	I.
Classical Course:				
Algebra, hours weekly,	2	2	2	—
Geometry, "	2	0	0	—
Scientific Course:				
Algebra, hours weekly,	2	3	3	—
Geometry, "	2	3	3	—

From Brown, E. E., *Secondary Education in the United States*, Albany, 1900, being No. 4 of monographs on education prepared for Paris Exposition, 1900, by the Department of Education.

simply teach the algebra and geometry side by side instead of in two halves of the year.

The teacher may well begin with a simple juxtaposition, using one text for algebra, another for geometry. The interrelations need not be forced; they will grow, and the teacher will be surprised to see how often one subject throws light on the other or can be used as material, when once they are taught side by side with the feeling that they *may* be interwoven on occasion instead of being kept separate and "pure," uncontaminated each by the other. The best results will not be reached on the first trial or the second, but with time the two subjects will be so thoroughly intergrown that it will seem a marvel that they could ever have been taught disconnectedly.

Co-ordination of Mathematics with Physics

A more radical and better change would require the co-operation of the department of physics, and would consist in teaching the co-ordinated mathematics throughout the four years, though with no larger total time; and the physics throughout the four years hand in hand with the mathematics, but with no change in its total time. This change would require some readjustment within the departments of mathematics and physics, but can be put into effect without any changes in the rest of the curriculum. The change in the mathematical work alone may well be made first, as it is the easier, and it will be a direct step towards the later reorganization of the work of the two departments.

An Example of Readjustment.

How slight a disturbance would be caused by readjustment of the mathematical work of a single school to permit simultaneous instruction in algebra and geometry is readily seen by considering a special example. Consider a school distributing the time allotted to mathematics and physics as follows:

First Year.	*Second Year.*
Algebra, 5 hours.	Plane Geometry, 5 hours.
Third Year.	*Fourth Year.*
Algebra, Solid Geometry, 5 hours.	Physics, 5 hours.

The same time might be distributed as follows, involving only redistribution of the work of the classes in mathematics:

First Year.	*Second Year.*
Mathematics, 5 hours.	Mathematics, 5 hours.
Third Year.	*Fourth Year.*
Mathematics, 5 hours.	Physics, 5 hours.

In its simplest and most conservative form this plan merely proposes a rearrangement of the work in algebra and plane geometry of the first two years, so that the easier parts of each subject are taken up in the first year, and the more difficult parts of each in the second year. This would be a cautious experiment, one concerning which the most conservative need feel no misgivings, and which can be undertaken without any reorganization whatever. All that is needed is a redistribution of topics to be taken up, which can be effected by an agreement (with proper sanction) between the teacher of first-year algebra and the teacher of plane geometry. The results of the experiment will point the way either to further experiment or to return to the old plan.

Ease of Adjustment.

The way in which the time of the first years would be distributed between algebra and geometry would depend upon circumstances. It should certainly *not* be of the type Monday, Wednesday, Friday, *Geometry;* Tuesday, Thursday, *Algebra;* or any distribution allotting precisely this time to algebra and that to geometry. It is an essential feature of the plan that algebra and geometry be not kept rigorously separated. Fragmentary work should be avoided. When a topic in algebra or geometry is taken up it should be advanced to a suitable extent, far enough to make a point, and not dropped at an unfavorable juncture merely to make a change from algebra to geometry, or *vice versa*. The ideal is to take up a *topic* neither as algebra nor geometry exclusively, but with the assistance of the methods of both. Thus addition would mean not only addition of numerical and literal data, but also addition of line segments. The sum of the angles of a triangle, of a polygon, would be accompanied by

Coherent Treatment of Topics.

algebraic exercises in which the angles are unknown. Square root would be associated with the Pythagorean theorem, etc.

To bring the different branches of mathematics into closer touch is not only the most feasible but also the most promising change possible in the secondary school curriculum in mathematics.

Mathematics and Physics.

The mathematical work could be enriched and made more interesting, and the work in physics also strengthened, if physics and mathematics could be taught simultaneously, the mathematical theory thus standing much nearer in time to its physical application than it does at present. With the co-operation of the department of physics the time might be distributed thus:

First Year.	*Second Year.*
Mathematics, 4 hours.	Mathematics, 4 hours.
Physics, 1 hour.	Physics, 1 hour.
Third Year.	*Fourth Year.*
Mathematics, 4 hours.	Mathematics, 3 hours.
Physics, 1 hour.	Physics, 2 hours.

Specimen Distributions of Time.

This distribution of the time is open to the serious objections against giving one or two hours weekly to a subject. The objections would not be quite so serious if the mathematics and the physics of any one year could be placed in the hands of the same instructor. On the other hand, the plan has the strong advantage of not allowing the mathematics to rust during the fourth year, and of teaching it at the time when the need for its use arrives in physics. The plan involves no more instruction on the part of either department.

Another distribution, concentrating the physics somewhat more, might be:

First Year.	*Second Year.*
Mathematics, 5 hours.	Mathematics, 5 hours.
Third Year.	*Fourth Year.*
Mathematics, 3 hours.	Mathematics, 2 hours.
Physics, 2 hours.	Physics, 3 hours.

Other arrangements will suggest themselves according to local needs.

Should Work in Mathematics be same for all Pupils?

The question may be raised whether there should be a difference in the work in mathematics according to whether the pupil intends to go to college, to technical school, or into active life. The general reply has been,[8] *No.* Mathematics aims to train such fundamental powers that its study is a requirement of practical education, not merely of college entrance regulations. It has been urged that there is a marked difference both in needs and in intellectual capacity between the pupils who go to college taken as a whole and those who do not. It may be questioned whether this difference exists in mathematics, or even whether both the need and the capacity for mathematics may not be greater among those who are not preparing to go to college.

[8] See Reports, Committee of Ten, Com. on Coll. Entrance Requirements.

CHAPTER XI

DEFINITIONS AND AXIOMS

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Definitions

What is a Definition?

IN mathematics a definition is simply an agreement making clear the precise meaning which is to be attached to a certain word, expression or symbol during the time in which the definition remains in force. A definition is usually a formal and permanent description, and the notion of the thing described should exist clearly in the mind before the formal and permanent description is set up. The purpose of the definition may be either to introduce a single word or symbol for that which up to the time had been described in longer phrases, or to secure more exact agreement in the use of a term already used loosely in common parlance.

Pascal's rules for definitions are often quoted:[1]

1. Make no attempt to define terms themselves so clear that no clearer terms exist in which to explain them.

2. Admit no uncertain or obscure terms without definition.

3. In definitions use only terms which are perfectly clear in themselves or which have already been defined.

Classification of Definitions.

For purposes of discussion, the notions usually defined by workers in mathematics may be classified under three heads: elementary notions, general terms and the others.

1. *Elementary notions.* The elementary notions are the simple ideas which are incapable of description in simpler terms. There evidently must be some such notions; there must be bottom somewhere. With these, which are really indefinable from the point of view of strict theory, may well be associated those for which a real definition could be given, but which are sufficiently simple to the pupil without formal definition. Terms of the class thus formed are straight line, unity, point, angle, direction, motion, number, quantity, etc. It is a waste of time, or worse, to discuss with pupils notions of this character. The profitable consideration of the fundamental notions can come only after experience in the subject and

[1] Pascal, *De l'Esprit géométrique*, 1655.

requires abstract thought. All that is needed in the secondary school is that the pupil have a working knowledge of what is meant by the terms. Let him philosophize in the university.

General Terms. At the opposite extreme stand general names, terms of classification, as, mathematics, algebra, geometry, proposition, theorem, scholium, etc. Here exact distinction is sometimes hard and always unnecessary, just as it is in some instances difficult to distinguish between animal and vegetable life, and still harder to frame a definition which shall infallibly make that distinction. It is difficult, for example, so to define "horse" that the description would fit every horse and nothing but a horse; yet in practice one would seldom mistake a horse for anything else.

It is a matter of little consequence whether a certain assertion is a proposition or a theorem, a scholium or a corollary, whether it belongs to algebra or geometry, to mathematics or to physics: the thing itself is what is important. A sufficient working knowledge on the pupil's part is obtained by using the terms; "this which we are studying is algebra"; he gets a sufficient idea of what a theorem is by having things presented to him labelled "theorem."

Other Terms. Formal definition, then, will be restricted to things which are neither elementary notions nor to be treated as such, nor yet terms of general classification. Here Pascal's canons of definition come into play. But they should not be formulated to the pupil. The essential thing always is that the pupil have a good working knowledge of the notion, and that he use correctly the term set apart by convention to designate the notion. The definition is a description of a thing, an idea. The thing itself should first be presented to the pupil; as he works with it he will see its properties, he will describe it on occasion, and will think of a name for it when he feels the need of a name. Definitions are an outgrowth of the work rather than the basis of it.[2]

In some cases the conventional name describes the object

[2] For contrary view see Report Committee of Ten.

of thought sufficiently well that the pupil will think of it (or its English translation, since our scientific names are unhappily usually in Greek or Latin), but in many cases the teacher must give the name.[3] The teacher should of course see to it that the definition is finally left in a correct form.

Redundant Definitions. This does not exclude redundant definitions such as pupils are quite likely to reach. The definition reduced to the minimum is the ideal, towards which the instruction tends; but perfection is not reached in a day, and while progress should be made, there may be occasions when it would be better to accept a redundant definition than to divert from the main line of progress the time, energy and attention needed to trim and polish the redundant definition. Errors, of course, cannot be tolerated.

Definitions are, in a sense, a variety of axioms. We shall see that it is pedagogically imperative, in view of the present state of knowledge on the subject of axioms, to work with a redundant body of axioms, making no effort to reduce them to the minimum. A rich body of definitions and axioms, of assumptions that seem sufficiently obvious, facilitates progress to the interesting parts of the subject, while undue insistence on technicalities in this respect retards progress and tends to dry up the pupil's interest. One who shares this point of view would not object, for example, to defining a rectangular as a parallelogram all of whose angles are right angles, although it would be sufficient to stipulate that one angle is a right angle.

Definitions not unalterable. A definition being an agreement of the sense in which a term is to be used is dependent upon the parties to the agreement. In the case of teacher and pupil the scope of the pupil's view should control the definition, not that of the teacher. For a child who knows no fractions, "number" means what the teacher might call "real, rational, integral number." It is not necessary to say *whole* number to a child who had not extended the number concept

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[3] Workman, in Spencer, *Aims and Practice of Teaching*, pp. 198, 199, gives a specimen lesson in eliciting definitions.

to fractions, or *rational* number to a child who had no idea of irrational number. A definition is satisfactory if it covers the case in hand, being complete from the pupil's point of view, though it may be partial from the teacher's. It must, of course, be generalized from time to time as the pupil's horizon widens. Unnecessary confusion may be caused by constantly inserting provisos to cover cases of which the child knows nothing. On the other hand, the extreme of inculcating a restricted definition so firmly that it is an obstacle to generalization is an error also to be avoided.

In practice it may be well to lay as little stress on definitions as is consistent with good working knowledge.

Definitions should lend themselves to Generalization.

While it is a mistake to seek complete definitions at the outset, while even teachers cannot be expected to know all that the mathematical world has labelled with one and the same term, still the definition should be good as far as it goes, and should lend itself easily and naturally to generalization. It should never be necessary to cast away a definition once adopted; for the same restricted field, the same definitions should hold for the advanced mathematician which he used as a child, though he may seldom regard that field otherwise than as a part of a larger field. As examples, some of the definitions of geometry, as circle, tangent, may be cited (see the chapter on Geometry).

Axioms.

There have been three principal views as to origin and character of axioms, namely, that they are:

Three Views as to Axioms.

1. *A priori* truths; self-evident truths (Kant).
2. Experimental facts (J. S. Mill).
3. Conventions (hypotheses, definitions).

The last is the view of modern mathematicians. An axiom is no longer regarded as a "self-evident truth," but simply as a statement accepted without demonstration to be the basis of the considerations in hand. There is from this point of view no *essential* difference between axiom, postulate, hypothesis. There are, however, differences in usage. The term *axiom* is usually used for a (simple) supposition fundamental to a large subject, while a similar supposition relative to a particular problem would be called a *hypothesis*. The term *axiom* usually connotes relative simplicity, so that it is at least not easy, perhaps not possible, to deduce the statement in question from other (and simpler) ones. The term *easy* relates to the individual, and may require quite a different body of axioms for a professional mathematician and a high-school pupil.

Mill's View.

The contention of Mill[4] that the axioms of geometry are merely hypotheses is now generally admitted by mathematicians, who no longer even insist upon the concession which Mill makes that "these hypotheses are real facts with some of their circumstances exaggerated or omitted." Mathematics concerns itself with the conclusions that follow from certain hypotheses, whether or not these hypotheses formulate facts either exactly or approximately. The conclusions of mathematics can be *applied* whenever facts are found to exist which correspond to a set of hypotheses of mathematics with such approximation that for the question in hand the differences may be disregarded. Thus mathematics reasons about circles quite unconcerned about the question as to whether or not circles actually exist or are conceivable. If in practice a plane curve is found such that the distances of its points from a fixed point are so nearly equal that we are ready to overlook the differences, mathematics states approximately the area enclosed by the curve. If an exact statement can be made as to the amount of the differences, an exact statement can be made as to the area. Thus, if we have a closed plane curve of which we can say that no straight line from the centre to any point of curve differs in length from one hundred feet by as much as one foot, then we can say exactly, that the area enclosed by the curve is greater than $\pi \overline{99}^2$, but less than $\pi \overline{101}^2$. (Of course similar considerations apply to the other points involved or implied, for example, the

[4] John Stuart Mill, *System of Logic*, Bk. II., Ch. V., par. 1.

"plane-ness" of the curve. The reasoning above has been centred upon the equality of radii, as sufficiently illustrating how the purely hypothetical conclusions of mathematics may be applied in practice).

It is only within quite recent years that accurate formulations of the axioms of geometry have been made. As an example, the axioms of Veblen[5] may be cited.

The Researches of Recent Decades.

The propositions brought forward as axioms by Veblen are stated in terms of a class of elements called "points," and a relation among points called "order." All other geometrical concepts, such as line, plane, space, motion, are defined in terms of point and order. In particular the congruence relations are made the subject of definitions rather than of axioms. The terms "point" and "order" differ from the other terms of geometry in that they are *undefined.*

Veblen's Axioms.

The axioms are twelve in number; they presuppose only the validity of the operations of logic and of counting (ordinal numbers).

Axiom I. There exist at least two distinct points.

Axiom II. If points A, B, C are in the order $A\,B\,C$, they are in the order $C\,B\,A$. The point B is said to "lie between" A and C.

Axiom III. If points A, B, C are in the order $A\,B\,C$, they are not in the order $B\,C\,A$.

Axiom IV. If points A, B, C are in the order $A\,B\,C$, then A is distinct from C.

Axiom V. If A and B are any two distinct points, there exists a point C such that A, B, C are in the order $A\,B\,C$.

Def. 1. The *line* $A\,B$ consists of A and B and all points X in one of the possible orders $A\,B\,X$, $A\,X\,B$, $X\,B\,A$. The points X in the order $A\,X\,B$ constitute the *segment* $A\,B$. A and B are the *end-points* of the segment.

Axiom VI. If points C and D lie on the line $A\,B$, then A lies on the line $C\,D$.

Axiom VII. If there exist three distinct points, there exist three points A, B, C not in any of the orders $A\,B\,C$, $B\,C\,A$, or $C\,A\,B$.

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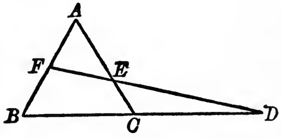
[5] See Bibliography.

Def. 2. Three distinct points not lying on the same line are the *vertices* of a *triangle* $A B C$, whose *sides* are the segments $A B$, $B C$, $C A$, and whose *boundary* consists of its vertices and the points of its sides.

Axiom VIII. If three distinct points A, B, and C do not lie on the same line, and D and E are two points in the orders $B C D$ and $C E A$, then a point F exists in the order $A F B$, and such that D, E, F lie on the same line.



Def. 5. A point O is *in the interior of* a triangle if it lies on a segment, the end-points of which are points of different sides of the triangle. The set of such points O is *the interior* of the triangle.

Def. 6. If A, B, C form a triangle, the *plane* $A B C$ consists of all points collinear with any two points of the sides of the triangle.

Axiom IX. If there exist three points not lying in the same line, there exists a plane $A B C$ such that there is a point D not lying in the plane $A B C$.

Def. 7. If A, B, C, and D are four points not lying in the same plane, they form a *tetrahedron* $A B C D$ whose *faces* are the interiors of the triangles $A B C$, $B C D$, $C D A$, $D A B$ (if the triangles exist), whose *vertices* are the four points, A, B, C, and D, and whose *edges* are the segments $A B$, $B C$, $C D$, $D A$, $A C$, $B D$. The point of faces, edges, and vertices constitute the *surface* of the tetrahedron.

Def. 8. If A, B, C, D are the vertices of a tetrahedron, the space $A B C D$ consists of all points collinear with any two points of the faces of the tetrahedron.

Axiom X. If there exists four points neither lying in the same line nor lying in the same plane, there exists a space $A B C D$ such that there is no point E not collinear with two points of the space $A B C D$.

Axiom XI. If there exists an infinitude of points, there exists a certain pair of points $A C$ such that if (s) is any infinite set of segments of the line $A C$, having the property that each point which is A, C or a point of the segment $A C$ is a point of a segment s, then there is a finite subset $s_1, s_2, \ldots {}_n$, with the same property.

Axiom XII. If a is any line of any plane α there is some point C of α through which there is not more than one line of the plane α which does not intersect a.

The first theorems proved by means of these axioms are statements which are ordinarily regarded as simple and accepted without even formulation.

For example:

Theorem 1. (Proved by Axioms II. and III.)

If points *A*, *B*, *C* are in the order *A B C*, they are not in the order *C A B*.

Proof. Since the points are in the order *A B C*,

1. They are in the order *C B A* (Ax. II.).
2. Hence they are *not* in the order *B A C* (Ax. III.).
3. Therefore they are *not* in the order *C A B* (Ax. II.); for if they were in the order *C A B*, they would by Ax. II. be in the order *B A C*, which is contrary to (2).

Theorem 2. The order *A B C* implies that *A* is distinct from *B*, and *B* from *C*.

Proof. 1. If *A* were the same as *B*, orders *A B C* and *B A C* would be the same.

2. But if points are in order *B A C*, they are in order *C A B* (Ax. II.).

3. Hence we should have points in order *A B C*, and *C A B*, contrary to Ax. III.

Therefore *A* must be distinct from *B*.

Other theorems are:

Theorem 3. Every pair of distinct points *A*, *B* defines one and only one line *A B*, and one and only one segment *A B*.

Theorem 4. Any pair of distinct points of a line determine it.

Theorem 5. If *D E* is any line, there exists a point *F* not lying in this line.

Theorem 6. Between every two distinct points there lies a third point.

Theorem 9. To any four distinct points of a line the notation *A*, *B*, *C*, *D* may always be assigned so that they are in the order *A B C D*.

Theorem 9 was an *axiom* in Hilbert's geometry, but Moore [6] showed that it can be *proved* from the other axioms. This fact serves to illustrate the difficulty which attends a

[6] Moore, On the Projective Axioms of Geometry. *Trans. Am. Math. Soc.* 1902.

serious effort to reduce the list of unproved statements to a real minimum.

Desiderata for a System of Axioms.

For a system of axioms which is irreducible, it is requisite that the set be (1) *complete*, that is, that it formulate explicitly all the axioms that are used; (2) *consistent*, that is, that no axiom contradicts the others; and (3) *independent*, that is, that none be deducible from the others. In elementary work, only the postulate of consistency can be retained. The example of a system of axioms given above makes it clear that it is not practicable to enumerate in elementary geometry all the axioms there used, nor can those used be restricted to such as are indemonstrable by means of the others. No proof even of the consistency of those used can be attempted. It is sufficient that they are valid in the concrete geometry of the world about us; and such special verification, by the way, is the mode of proof used by Hilbert and Veblen in establishing the consistency and the independence of their axioms.

Effect on Elementary Teaching.

The profound study of axioms should have equally penetrating effect on elementary teaching, — *not* a still more exacting treatment of axioms in elementary work than heretofore, but just the reverse.[7] A few decades ago teachers and text-book writers fondly imagined that they were formulating in words all the axioms

[7] "In the domain of advanced mathematical research, investigations on the foundations of our science, its hypotheses, or, as we prefer to say, its *axioms*, are at present in the foreground of interest. It is possible that some enthusiastic mathematician may undertake to carry over the results of this investigation into the instruction of the schools. If this is cautiously done by way of allusion in the highest class before pupils whom the teacher has succesfully accustomed to more abstract processes of thought, no one will blame him. But there are authors who begin their books, intended for school use, with detailed and abstruse formulation of new systems of axioms. This may be interesting scientifically, but it will meet no success with our teachers. The German school as it has developed, and has had to develop in the last decades, unconditionally rejects such attempts. Its first principle is everywhere to adapt

[illegible] citing them by number whenever used, that the axioms so formulated were self-evidently true, and that every one was incapable of demonstration by means of the others. The researches of recent times have effectively dispelled this illusion. It is now seen that many axioms are constantly used without formulation — simply tacitly taken for granted; that others can be replaced by much simpler ones; and that the whole subject is one of extreme delicacy, known in its completeness by only a few score men, and requiring such thorough command of the most subtle logic that it has been possible for leaders in the researches, mathematicians of world-wide eminence, to reach and to publish inaccurate results.

The axioms and theorems above are cited to show how few and simple are the statements from which alone all the others can be proved, and how futile is the campaign that is being waged, even to this day, against "taking nothing for granted that can be proved." The set of axioms given illustrate effectively the depths to which one must dig to build a really solid structure of "Axioms." Let the difficulty, delicacy, and depth of the subject thus made manifest help to assuage the qualms of conscience of those teachers who feel scruples lest by deviating from the traditional form of Euclid and his followers they may accidentally give a redundant definition, or may make use, without proof, of some sufficiently obvious statement which might be proved by means of what has gone before. The recent researches on axioms explode, once for all, all hope of teaching the child a logically perfect geometry in which all the results are deduced from a set of irreducible first principles, and with it goes the only justification for adhering longer to the semblance of such a rigorous deductive system.

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the work to the powers and the natural interests of the pupils. The model of Euclid, which has so long been used to support the contrary procedure, is misleading. Every edition of Euclid should be prefaced by the statement that the great author of the 'Elements' certainly did not write for boys." — F. Klein, *Verhandlungen d. Breslauer Natur-Forscherversammlung*, 1905, p. 38.

As used in secondary-school work, the term *axiom* in addition to what has been said connotes a "moral" certainty that the statement made is really true in the geometry of the world about us. For the high-school pupil, only such statements should be treated as axioms as seem to him to have a certain *validity* independently of himself, leaving to much later years the generalization from this "actual" validity to the purely hypothetical validity which characterized the notion of axioms in the mathematical world of to-day.

The Term Axiom in the Secondary School.

For the high-school pupil, then, the axiom is, as of old, a *truth* so obvious that it requires no demonstration to establish its validity.

This being the case, the sound pedagogic treatment seems to be simply to *use* the axioms when needed, perhaps without formulation, simply with a tacit or open "of course."

What should be the Treatment of Axioms?

This has been done by every one as to many axioms of geometry;[8] *e. g.*, "Between every two points on a straight line lies a third." Very few of the axioms formulated in geometry are really axioms of geometry at all, but of quantity in general, or are fundamental laws of thought.

It will be noted that the body of axioms for any subject as presented to pupils will usually be *redundant*, that is, it will contain statements which could be proved as consequences of the others. This is not a fault, though, of course, *contradictory* axioms would be.[9] In the latter case, show the pupil the contradiction either theo-

The System used will be redundant.

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[8] "If it were not for that bugbear of an examiner I am not sure that I should not advise the almost total omission of the axioms. It is not as if they were complete. They are simply an early attempt to formulate the laws of thought. Euclid assumes several which he does not specifically mention." — Workman, in Spencer, *Aims and Practice of Teaching*, Cambridge, 1897, p. 200.

[9] "When it comes to the beginning of the more formal deductive geometry, why should not the students be directed each for himself to set forth a body of geometric fundamental principles, on

retically or by concrete examples. In the former, let him go on, and return later when he sees the possibility of deducing his axioms from others. The normal development will thus lead to a gradual reduction of the list of axioms, until a really irreducible list may finally be reached by a Hilbert or a Moore.

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which he would proceed to erect his geometric edifice? This method would be thoroughly practical and at the same time thoroughly scientific. The various students would have different systems of axioms, and the discussions thus arising naturally would make clearer in the minds of all precisely what are the functions of the axioms in the theory of geometry. The students would omit very many of the axioms, which to them would go without saying. The teacher would do well not to undertake to make the system of axioms thoroughly complete in the abstract sense.

"'Sufficient unto the day is the precision thereof.' The student would very probably wish to take for granted all the ordinary properties of measurement and of motion, and would be ready at once to accept the geometrical implications of co-ordinate geometry. He could then be brought with extreme ease to the consideration of fundamental notions of the calculus as treated concretely, and he would find those notions delightfully real and powerful, whether in the domain of mathematics or of physics or of chemistry." — Moore, *Presidential Address*.

CHAPTER XII

THE TEACHING OF ARITHMETIC

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The Aim in Teaching Arithmetic

As purposes of the teaching of arithmetic these may be mentioned:

Special Purposes of the Teaching of Arithmetic.

1. To teach the child the mathematical type of thought.

2. To arouse his interest in the quantitative side of the world about him.

3. To give accuracy and facility in simple computations.

4. To impart a working knowledge of a few practical applications of arithmetic.

5. To prepare the way for further mathematics.[1]

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[1] It has been proposed to regard as chief purposes of study of arithmetic not —

1. Attainment of knowledge of useful processes;

2. Mental discipline;

but the arousing of a deep interest by study of such materials as

a. Correspond with the child nature;

b. Identify the child with actual life. — Smith and McMurry, *Teachers' College Record*, New York, March, 1903.

Arithmetic as a Type of Thought

To attain the General Ends of the Study of Mathematics.

What has been said as to the purpose and value of the study of mathematics in general applies, properly interpreted, to arithmetic with as much force as to other branches of mathematics. The primary object is to give knowledge of a certain type of thought, and skill and ease in its use. It need not be said that in comparison with later teaching the immature and rapidly growing mind of the child requires very important differences in method and mode, but these differences relate to the manner of attaining the ends rather than to the ends themselves. Throughout all mathematics, from the first numbers lisped in the nursery to the aged mathematician's last sigh, the chief end of mathematics is thought, not routine, — *natural* thought, exercising the powers of the thinker in an unforced and interested manner, not a forced and convulsive struggle for what is beyond grasp.

True, an important part of the work in arithmetic is to teach certain simple processes,[2] and to give sufficient drill to

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[2] For the ordinary purposes of non-technical daily life we need little of pure arithmetic beyond:

1. Counting, the knowledge of numbers and their representation to billions (the English thousand millions).

2. Addition and multiplication of integers, of decimal fractions with not more than three decimal places, and of simple common fractions.

3. Subtraction of integers and decimal fractions.

4. A little of division.

Of applied arithmetic we need to know:

1. A few tables of denominate numbers.

2. The simple problems in reduction of such numbers, as from pounds to ounces.

3. A slight amount concerning addition and multiplication of such numbers.

4. Some simple numerical geometry, including the mensuration of rectangles and parallelopipeds.

Enough of percentage to compute a commercial discount and secure and to keep up accuracy and speed in their use, and to do this with continual reference to the conditions under which the child or the adult is likely to use them. Indeed, this may well be the apparent reason for the study from the child's point of view. But the teacher who works for no other end than this will miss a thousand opportunities to invigorate the pupil's power of thought. All the practical usefulness of arithmetic can be attained without sacrificing its value in cultivating the habits of observation and reflection.

Teaching Processes.

Routine work or parrot-like repetitions need not be tolerated at any stage, even the earliest. It *is* possible to bring arithmetic within the comprehension of the child, to make it reasonable, natural, and interesting to him.

Three types of thought are prominent in arithmetic:

Types of Thought in Arithmetic.

1. To understand statements.
2. To observe properties.
3. To make inferences.

The subject-matter concerning which this thinking is done is the very simplest — the elementary relations of number and form. This simplicity is the glory of our subject. Arithmetic is a science of observation, but the observations are so much a matter of course that it is not ordinarily classified with the sciences of observation. What teacher has not often heard the exclamation, "Oh, I did n't *see* that!"

The Child's Power to reason.

Don't think that the child is too young to reason. Appeals to reason, asking the child to make inferences, to draw conclusions, have been discouraged by some on account of the undeveloped state of the child's mind. The reasoning power is supposed to be latent and the mind occupied solely with the unreasoning reception of sense impressions. No sadder mistake could be made. The everlasting "why?" of the child amply refutes this

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the simple interest on a note. — Smith, *Teaching of Mathematics*, p. 21.

See also *Report Com. Coll. Ent. Req.*, pp. 21, 138–139.

assertion. His power of reasoning is active, alert, *irrepressible.* But instead of being nourished, it is too often fed on husks; instead of the *thing*, he received the *formulation.* If only the natural cravings of the child's mind were heeded, the early years could be made much more fertile, and if only we were not so anxious to get him into our adult abstractions, he would even at a very early age actually (though informally) make his own abstractions. The child-mind is very logical, and those who have much to do with children will testify with what pitiless and impeccable logic the children hold their elders to the legitimate consequences of their previous statements or deeds. The child who says "I goed" has gone through an elaborate process of induction and deduction; he has observed many instances, he has inferred a general rule, he has applied it. He has passed from the concrete to the abstract, and back again to the concrete. There is no flaw in his logic. It is the English language that is illogical. And it will be a long while before he will meet a more difficult piece of reasoning in arithmetic than that which he has carried through correctly here with no conscious effort. The difficulty in teaching the child the inferences of arithmetic lies not so much in the subject itself as in the form in which they are presented. The child mind is not ready for the cut-and-dried formulation which the trained adult mind finds most satisfying. The little mind is constantly reasoning, and reasoning correctly, easily, spontaneously, but it is not able to attach the formal labels. The child is interested in the result and not the process.

The Child is logical.

A boy asked his mother to help him with a problem. She did so, but he did n't think it was right. When the father came home he was called on and gave the same solution. Still the boy was not satisfied. Next day he came home triumphant. "I told you you did n't work that problem right; you left out two 'hences' and a 'therefore.'" The difficulty of arithmetic lies largely in the "hences" and the "therefores," in the abstractions and set forms, so foreign to the child mind.

Beginning with suitable concrete cases of a character to interest the child, he will, informally but really, see clearly the general laws and apply them correctly, if only we can be content to keep things concrete and informal, and can restrain our impatience to have him reach formulations in the traditional words of past centuries.[8] His powers of abstraction will grow, will take firm root in his nature, if we refrain from constantly digging up the plant to see how the roots are getting on. Let the plant grow, until by its own force it bursts the concrete pot, which may, of course, be intentionally made thin and perhaps scraped a little when it begins to give way. Leaving the metaphor, the power of abstraction will surely prove all the stronger, if not formally exercised too soon, if never burdened with tasks that strain. It may be urged that the early years are the years of acquisition of vocabulary, and that the child will then learn names most easily. This is no doubt true, and there is no reason why the child should not learn the names as soon as he knows the things. When he learns the name, he should at once learn the name that is to serve permanently, and not some makeshift which is to be discarded later (*e. g.*, *oblong* for *rectangle*). But the criterion for the giving of the name is always the having the *thing*, and the thing should never be brought forward merely for the sake of giving the name.

Abstract Arithmetic difficult.

When arithmetic is taught in this spirit, it is needful from the very outset that the pupil always work with a purpose; that is, that he first collect his wits and decide what is to be done before doing anything. Do not think a child is too young to have an intelligent purpose. If in any particular case it is not easily possible for the child to think out his plan, that is *prima facie* evidence that the

Planning Work.

[8] "This little brain is full of curiosity; it seeks to know and to discover rather than to comprehend; instead of furnishing food for this curiosity, we weaken it, we discourage it; we impose a sort of intellectual obedience on this mind, instead of favoring, of perpetually inciting this initiative with which it has been so highly endowed by nature." — Laisant, *L'Éducation*, p. 27.

question has either not been put sufficiently simply, or is essentially too hard. To go faster than the child can *think* is to go *too* fast, and to descend from education to mechanical cramming.

The execution of any piece of mathematical work consists of several parts, none of which may be neglected.

1. *Grasping the problem*, getting a clear idea of what is known and what is required.

2. *Planning the work*, deciding how to ascertain the desired information from the known facts. The first plan made may not be successful, but there should always be an intelligent plan.

3. *Execution of the plan*. This is carried so far that one is convinced either that the plan will not work (in which case he tries another), or until the result is attained.

4. *Testing the result*. Compare the result with the data of 1, and make certain that one has really done what he set out to do.

Pedagogic Suggestion.

In accordance with the sound pedagogic practice of separating difficulties, exercises on each of these heads may profitably be taken up separately. Thus, lists of problems may be gone over in the class simply under 1, the pupil reading the problem and formulating clearly in his own words what is known and what is to be found. When this can be done well, step 2 may be taken up. Throughout the work in arithmetic, and indeed later, many problems should not be carried further than steps 1 and 2.

In very simple problems intended chiefly for numerical drill, the plan may be so obvious that it would not be worth while to formulate it, but there must always be a plan. Never tolerate haphazard work.

The importance of knowing what is given and what is wanted cannot be over-emphasized. Pupils are too prone to plunge headlong into some form of computation without having held the data up clearly before their own mental vision.

The reasoning needed for the arithmetical processes is simple enough in itself, and the child will perform it without difficulty, provided he has some motive for doing it. A boy that I know, aged five, does not tire of counting the jackstraws he has captured, but he takes little interest in counting for me in the abstract. Soon he will find out for himself, or appreciate when told, that he can count them more rapidly by twos, etc. As the child grows older his range of interest widens: games, all his own motor activity, the life about him and that of which he learns through reading and in his study of history, geography, and nature. Arithmetic will not be dull when it is needed to solve problems which are interesting in some other connection.

Child needs a Motive.

Arithmetic and Nature

In arithmetic, perhaps more than anywhere else, it is necessary to arouse and hold the child's interest, and this can be done most effectively by letting the work in arithmetic spring out of his own activities, or stand in close relation to them and to his experiences and observation of the world about him.

The Quantitative Side of Nature.

Stuart Mill urges[4] that "there are no such things as numbers in the abstract; all numbers must be numbers of something." Without entering upon discussion of this point, no one would dispute the concrete connection of the numbers used by children. In consequence, devices have long since been used to make numerical work concrete to the child. At the outset he has been given objects to handle, he has been taught to add three apples and eight apples rather than three and eight. But still many children find the subject dry and abstract. To-day emphasis is being laid on the fact that the term "concrete" is *relative*, and that what seems simple and concrete to the adult may appear quite otherwise to the child. It is pointed out that $\frac{2}{3}$ of $4\frac{5}{6}$ bushels of wheat is in reality no more concrete than $\frac{2}{3}$ of $4\frac{5}{6}$; that it is not sufficient to abandon the traditional problems of bygone centuries; that it is necessary to interest the child by bringing arithmetic into

The Concrete.

[4] Logic Bk. II., Chap. VI., par. 2.

close touch with real life as he sees it. The problems are made to relate to what comes within the range of *his own* experience, to what he actually sees, or at least can easily understand, and, best of all, to his own activities. Only thus is the work really concrete.

A large amount of interesting and valuable quantitative material may be readily gained from the pupil's own activity in connection with nature study. Current works on nature study do not as a rule bring the mathematical side of the work into the foreground, but there is an important background of mathematical relations which the teacher will have little difficulty in seeing and bringing out, and with perhaps slight modifications nature study can be made to bristle with arithmetical and geometric questions.

Sources of Material.

But it is not necessary that the child engage in nature study in order to obtain access to the domain of interesting applications of arithmetic. Simple measurements and observations can be made by the pupil in little time and without apparatus other than the commonest articles within the reach of every one. Data for local problems can be obtained by simple observation and counting, — as the average number of houses per block on certain blocks; the percentage of the whole number which are of wood, of stone, of brick; the number of stores in a certain region; what fraction are dry-goods stores, groceries, etc. The number of persons who pass your house in five minutes; what per cent are men, women, children? At the same rate, how many would pass from 7 A. M. to 6 P. M.? At this number daily how many would pass in a week? a year? What part of the total population would this be? What per cent? etc.

Such a chain of problems gives occasions to put into practice many of the important processes of arithmetic. The results may not be of interest to the world at large, but they will decidedly be of interest to the child, because they utilize and expand facts he has furnished himself.

Local conditions will provide any number of similar possibilities. The water that fell in yesterday's shower, the rate of flow of the local river (if possible observed by the pupil himself by noting the time in which an object floats through a known distance), the number of cubic yards of snow that must be shovelled from the sidewalks of a certain block (of your home, of the schoolhouse) to remove yesterday's fall (the pupil measuring the depth, and the needed frontages if not already known).

Local Problems.

These are a few random instances showing how local occurrences may be used arithmetically. There are few issues of the local newspaper that will not furnish in advertisements as well as reading matter interesting facts that can be made the basis of arithmetical work. With but little more difficulty facts from local geography, local and familiar industries, current prices, local traffic, municipal administration and social conditions may be secured for the class in arithmetic. The interest will be greatest if the data are secured by the inquiry and observation of the pupils themselves.

As the pupil's horizon widens he will become interested in the industries and condition of his State and of the nation, especially in connection with the study of history and geography, commercial and political. The text may be expected to furnish problems of this character, but it cannot give problems of strictly local application. Its more general problems will, however, serve as types, and its specific directions may be applied also to local data; the hints easily to be read between the lines of a good text will be of much help to the teacher, provided he is really looking for such help.

On the streets of German cities it is a common sight to see classes of school children inspecting objects of interest under the guidance of the teacher, who points out and explains to them the important buildings, their uses and history, and the evidences of municipal life and organization that may appear. The history and import of monuments, the lives of men whose statues are visited are touched on according to circumstances. Country walks also have educational features of equal value. Such walks may be made most interesting and instructive, combined with due preparation for what is to be seen and with subsequent utilization of what was seen.

There is a fertile field for similar walks in our cities and country-sides, but I do not recall ever having seen or heard of one being made. The object of such a walk would of course be fixed and explained beforehand. It may be historical, geographical, botanical, zoölogic, economic, but it is also quite possible to make excursions with arithmetical purpose. Among the chief aims would be to teach the pupils to see what they have been studying in arithmetic exemplified in the world about them, and also to accumulate data for further class use.

Arithmetical Excursions.

This might be achieved, for example, by having the children look for and record in note-books instances of occurrence of the various geometric forms they have been studying (triangles, rectangles, trapezoids, spheres, cylinders, etc.), or some designated particular one of them. Occasionally numerical data can be secured by inquiry, counting, or measurement. If the post office is visited, the number of boxes might be calculated (without actual counting of all), the rental ascertained, number not rented, etc., and later, problems made and solved about the facts. Perhaps, if requested beforehand, the postmaster would be willing to meet the party for say five minutes and give them some numerical facts concerning the business transacted by the office, the children and teachers noting the numbers and making them the basis of later work. If feasible to have the party shown through the post office, the interest would be enhanced still further.

Equally fertile might be similar visits to library, court house, factories, warehouses, markets, bank, stores, water works, gas plant, electric light plant, street railway barns and offices, railroad station and offices.

Among these and other possibilities the teacher would have little difficulty in finding a variety of places where a visit would be welcome, and some one would be deputed to give the party information and numerical data concerning the work done. The facts accumulated on the walk and visit might easily be the basis for a week's work in arithmetic, and would be incomparably superior to any other work in arithmetic in variety and interest. Even if half a day should be requisite for the excursion, it would be time well spent.

The attempt has been made to have pupils of the eighth grade actually perform some experiments in physics and make the consequent calculations.

Experiments in Physics.

The following is such a list of experiments in physics now used in the grammar schools of Cambridge, Mass. It has been kindly furnished me by Principal Frederick S. Cutter.

1. Volume of a solid by overflow of water from a vessel filled before the immersion of the solid.
2. Weight of a cubic centimeter of wood by measuring and weighing a block of rectangular or other convenient shape.
3. Weight of water displaced by a floating body compared with weight of the body.
4. Demonstration of the principle of Archimedes.
5. Specific gravity of a solid that sinks in water.
6. Specific gravity of wood by immersion with a sinker in water.
7. Specific gravity of wood by flotation.
8. Specific gravity of a liquid by specific gravity bottle.
9. Specific gravity of a liquid by loss of weight of a solid immersed therein.
10. The straight lever.
11. Experiment upon the centre of gravity of an irregular body, and the influence of its weight when it is used as a lever.
12. Levers of the second and third classes.
13. Force exerted at the fulcrum of a lever.
14. Laws of the simple pendulum.
15. Images in a plane mirror.
16. Index of refraction of glass.
17. Focal length of a lens.

As to the effect of introducing such problems Mr. Cutter says:

"I think the child's attention is largely taken up with the reasoning involved and the accurate computation rather than with the scientific truths which are presented in the problem."

This verdict means that as a first course in *physics* the work is not a success: such value as it may have is as an application of arithmetic. Under favorable conditions the teacher of arithmetic may be able to utilize some of these or similar experiments to give an addititional element of interest to the work.

What has been said as to the concrete source of problems must not be misunderstood. It is not meant to underestimate the need of substantial work on the arithmetical processes in themselves. But this side has not been emphasized in the discussion, because it has not seemed to need it.

Processes themselves must be studied.

It is quite evident, however, that all that has been urged as to the *source* of the problems will not tell *how to solve* them. The hope is simply that problems of the character indicated will interest the pupil and make him willing, even eager, to learn how to solve them, and thus transfer to the processes of arithmetic an interest which they in themselves rarely arouse.

Computation

The degree of skill in computation needed has been well defined thus:

"1. Ability to count infallibly objects occurring irregularly, up to two or three hundred; say, for example, packages of tickets or checks, dots upon a piece of paper, persons in a small audience-room, etc.

Accuracy and Facility in Simple Computations.

"2. The ability to add without the possibility of a mistake columns of figures such as would occur in an ordinary saving's-bank deposit-book or housekeeper's pass-book.

"3. The ability to add two numbers each below a hundred, or to subtract the less from the greater, rapidly and without recourse to pen or pencil.

"4. The ability to multiply, on the slate or blackboard, one number of a moderate length by a small multiplier, or to divide it by a small divisor.

"5. The ability to compute simple interest on moderate sums at even rates per cent for round periods.

"6. The ability to work simple examples in 'reduction' involving the use of the American table of weights, measures, and moneys.

"If every boy and girl on leaving grammar school at fourteen or fifteen had reached this stage of attainment, the public schools would have fairly done their duty by them so far as the practical uses of arithmetic are concerned.

"Schools have not done this — pupils could not add or multiply, subtract, divide, or even count with accuracy.

"It is difficult to imagine a greater wrong short of permanent injury to health that can be done to a child than to send him into the world to earn his living without the ability to conduct numerical operations accurately and with reasonable facility. Employers have literally no use for boys who make mistakes in number. Such a failing offsets the best training otherwise of mind and hand. In a store or a shop or factory, or on a railroad, a lad who cannot set down figures and add them right every time is little better than a cripple."[5]

Accuracy and Speed.

Accuracy and a fair amount of speed in the performance of simple arithmetical operations are demanded by practical needs, and sufficient drill to insure this is indispensable. That this drill should be chiefly on the simplest and easiest combinations and not carried on into more complex forms until the simplest are well mastered, has already been pointed out.

It may be added that *all* problems of arithmetic furnish incidental drill. A sixty ride ticket between Washington and Baltimore costs $15.45; how much is this per ride? is just as as much a drill problem as $15.45 \div 60 = ?$ Problems of the former type involve the same drill, and an additional element of interest, especially if the facts used have been ascertained by the pupil or are known by him as real.

Drill Problems.

It is a mistake to think that drill problems must be uninteresting or abstract. A problem may be called primarily a drill problem so long as the arithmetical work to be done is apparent to the pupil without delay. Problems which must be studied in order to plan the cal-

[5] Walker, *Discussions on Education*, p. 221.

culations, give incidental drill, but drill is not their primary object.

In view of the constant incidental drill in all problems, and of the large mass of concrete problems which can be worked just as rapidly as the corresponding abstract problems, it is evident that the need for problems of the last class is to a considerable extent obviated. If the pupil could get all his drill while working problems which interest him in themselves, the effect would be as happy as when he get his physical exercise by romping in the school yard rather than by mechanically working with pulley weights. This can perhaps never be done, but it is usually possible to place the abstract drill for mechanical facility and accuracy late in the treatment of the topic, and little mere "drill for drill's sake" should be needed.

There are two ways of learning: by routine drill, by mechanical repetition, on the one hand, or by content and interesting associations on the other. The child learns things in which he is interested by once telling. He need not be told a second time when the circus is coming. Why should a thousand repetitions be needed in arithmetic?

Practical Applications

Working Knowledge of a few Practical Applications.

In teaching practical applications of arithmetic the object is not so much to anticipate all possible applications that the pupil may have occasion to make, as to develop the *power to apply*, and to show how indispensable arithmetic is in every-day life. A few applications thoroughly mastered in form and in spirit, will be worth more than ten times the number simply stored up as facts.

Arithmetic and Business.

That the arithmetic taught in the schools should be available to the boy when he finds employment afterwards, and that the topics taught, and the manner of their teaching, should be such as to give the pupils the best average preparation for the probable arithmetical demands which they may have to meet later, seems a truism.

It is not so certain, however, that this end would be attained best by actually teaching the technical details of processes of arithmetic as used in various businesses.

Usage varies so much with the needs of different businesses, and customs change so rapidly with time, that though the pupil is taught scrupulously according to the present-day usage, he may find quite a different usage in vogue when he comes into practical business ten years or more from now. It is better to teach the unchanging fundamental principles and operations and train to adaptability in applying them wherever needed,[6] than to attempt to give a semi-professional training which may be out of date by the time the opportunity comes to put it into practice.

It would thus seem that while problems from practical and commercial life as the child sees it, and from the sciences, if within easy reach of his understanding and interest, cannot be used too freely, complicated and strictly technical business or other problems should be avoided. They cannot be adequately presented at school but they will be understood with ease when needed if the work in arithmetic has developed the power of grasping conditions and applying the elementary operations.[7] Whatever problems are used should be in harmony with actual practice. As to subject matter, the line may

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[6] "The easiest way for all currencies is the use of conversion tables, but a conscientious superior will not allow a clerk to use these before he is thoroughly familiar with all the calculations. When he thinks he knows it all, somebody instead of ordering a certain amount in foreign money will want to remit the equivalent of $19,385.65 to Paris: if the young man has his unlucky day, the manager will fix the rate at $5.18\frac{3}{4}$ less $\frac{3}{32}$ per cent; mopping of the brow will not help, but if he did such examples at school, he will smile and ask for something harder." — Kretz, *Banker's Magazine*, October, 1901, p. 707.

[7] "The fact that the arithmetic of business is pre-eminently the arithmetic of common sense should not for a moment be lost sight of in drilling classes in this branch of our schools." — N. E. A. Com. Report on Business College Course. *Report Com. Educ.* 1898–1899, Vol. II. p. 2163.

perhaps be drawn by including only such problems as average citizens might have occasion to use without following any particular trade or occupation. (This may include problems *about* many trades and occupations, but only such as would arise in the experience of those not following the trade or occupation in question.)[8]

Preparation for Later Mathematics.

To Prepare for Further Study of Mathematics.

Even though relatively few pupils study mathematics beyond the arithmetic of the grades, one of the purposes of the study of arithmetic may nevertheless well be to prepare the way for such study; this not for the sake of the few who go on, but because when taught from the broader point of view, the instruction will be best for even those who do not go on.

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[8] An excellent idea of what the business world believes it has the right to expect from schools is given in Still's report to the N. E. A. 1900, digested from six hundred replies representing fifty-seven different occupations and lines of business.

Mechanical Aids: —

1. Importance of decimal point in business.
2. Necessity for legible figures.
3. Accuracy and speed.
4. Use of interest and discount tables and graded schedules.
5. Use of cash registers and arithmometers.
6. Fractions like $\frac{8}{9}$, $\frac{14}{17}$, $\frac{19}{28}$, etc., should receive scant attention.
7. Importance of "short cuts."
8. Value of teaching multiplication table through 20×20.
9. Importance of familiarity with English money.
10. Business forms, such as checks, notes, receipts, statements, and price lists should be understood by all pupils of the seventh year and upward.
11. Constant repetition and drill in the use of the four fundamental processes.
12. Processes of Solution. (In reply to question as to differences between processes of school work and outside world.) On the whole, no great differences. (In interest, the only case, find *interest* and *amount*: facility in the four fundamental processes and approximate estimates important.)

This purpose may be reached: —

a. By so teaching the subject that the attitude of mind gradually grows to the broader view demanded in the next following mathematical subjects.

b. By developing the roots of following subject matter (without forcing) as they present themselves naturally in connection with various phases of arithmetic.

c. By so formulating what it is necessary to formulate (definitions, rules, descriptions of processes), that they lend themselves naturally and easily to subsequent generalization.

d. By *not* teaching that which must be undone later.

e. By omitting from arithmetic those topics which are better taught later.

To achieve these ends demands a wide range of mathematical knowledge on the part of writers of texts in arithmetic, and at least considerable range on the part of the teachers.

Simplifications

What Simplifications are needed in Arithmetic.

A comparison of the curricula and text-books in use to-day with those of fifty or even ten years ago, will show that considerable simplification has recently taken place. Circulating decimals, equation of payments, present worth, various rules for partial payments, gauging, and tonnage have generally disappeared. Cube root is seldom found now-a-days, and the problems taken up in denominate numbers are much less complicated. It is fortunately seldom that problems are now found like: "Express a long ton avoirdupois in oz. troy. Divide 7 mi. 62 rd. 3 yd. 2 ft. 7 in. by 1 mi. 84 rd. 2 yd. 1 ft. 11 in." Recent reports recommend still further simplifications.[9]

Processes that were valuable a few centuries ago have been superseded by better ones in the march of time. In the twentieth century, long additions are done by machinery, fractions are commonly decimal fractions, and consequently the process

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[9] See Reports of Committee of Ten; Committee on College Entrance Requirements.

of least common denominator is seldom needed; square and cube roots are extracted in a twinkle by logarithms; commercial applications are rapidly changing, and methods of algebra are freely used whenever they secure results more directly or easily than those of arithmetic.

The progress of our age and the needs of the day permit marked omissions in the subject matter of arithmetic, even as it was taught during the last decades of the nineteenth century. Some of these omissions are:

Some Omissions proposed.

1. *G. C. D. or L. C. M. of large numbers otherwise than by factoring.*

These processes are needed only in the reduction of fractions to a common denominator. In consequence of the common use of decimals the non-decimal fractions now chiefly used are those connected with our non-decimal measures. The denominators of all of them are readily handled by factoring.

2. *Fractions with large or unusual denominators.*

If the denominators 2, 3, 4, 6, 8, 12, 16, 20, 24, 32, 36, are mastered, the pupil need have no fear of trouble with fractions in actual practice. Fractions like 17ths or 29ths have little practical significance. Of course, the pupil must know what such fractions are, but they need not be emphasized. The pupil who can handle 24ths and 32ds will have no trouble with 26ths or 33ds if he should ever have to use them.

3. *Compound or complex fractions* as a special topic may also well be omitted. If the idea is mastered that $\frac{3}{4}$ means $3 \div 4$ there will be no difficulty in seeing that $\frac{\frac{3}{7}}{\frac{4}{9}}$ means $\frac{3}{7} \div \frac{4}{9}$; and problems in this notation will readily be understood if ever they are met later.

4. *All measures not actually in use in the community at large.*

Measures pertaining to special trades and occupations and which the general public has little occasion to use should be excluded. For example, Troy weight, apothecary's weight, apothecary's fluid measure, weights for precious stones, surveyor's measure, etc. The less common denominations should receive no more emphasis than their actual use in the pupil's community demands. It will be sufficient for most pupils if they know the meaning of gill, mill, rod, and these measures need be given no prominence in class-room drill. Of the measures taught only the customary combinations should be used. It is only in the school-room that a 2 gal. 2 qt. pail or a 12 yd. 1 ft. 6 in. building lot are ever heard of; in common parlance they are 10 qt. pail and 371 ft. lot. The greatest simplification in the matter of weights and measures will come only when the metric system comes into full and exclusive use.

5. *Reductions of decimals to common fractions and decimals beyond thousandths* should receive little emphasis.

6. *Circulating decimals.*

7. *Square root and cube root except by factoring.*

The proper place for these is in algebra. If the process of square root seems needful in the eighth grade, the rule may be given, but it is better not to try to explain it. It is too difficult for the child at that age.

8. *Profit and loss as a separate subject.*

This is nothing but a simple application of percentage, and there is no reason why it should be made a subject by itself. Some problems requiring the percentage of gain or loss are sufficient.

9. *True discount.* Bank discount has taken its place entirely.

10. *Partial payments* in the form of State rules and irregular endorsements. Modern methods permit no advance payments unless stipulated in the agreement. Such stipulation usually takes the form that money may be prepaid "on interest days" — *i. e.*, on the days when interest falls due, usually semi-annually. Payment in specific instalments, the modern partial payment, deserves attention.

11. *Equation of payments.*

12. *Partnership*, except very simple illustrations leading to an explanation of the modern stock company.

13. *Compound proportion.* Simple proportion is important, but is best treated as an equality of two fractions. The terms "inversion," "alternation," "composition," are quite unnecessary and merely confuse matters.

14. *Compound interest,* except a few problems to illustrate meaning of compound interest and the way in which it would occur in modern practice. The problems should be solved by use of a compound interest table.

15. *Business problems which do not conform to the usage of the day.*

16. *Large numbers and exercises involving many numbers* should also be excluded as a rule.

Every arithmetical operation is made up of a series of operations with very small numbers. For example, all addition is simply repeated addition of a number less than 10 to one less than 100; multiplication is a compound of addition and multiplication of two numbers less than 10; operations with fractions are combinations of the four fundamental operations with integers.

When once the fundamental operations with small numbers are thoroughly mastered, the same operations with larger numbers offer no new difficulty except that of repetition, but to impose the strain of manifold repetition when the single step is itself made with difficulty and uncertainty, is to invite failure.

The very young mind has enough to do to master these processes singly and then in combinations of a very few. To give long problems in addition, for example, is a grievous physiological and pedagogical error. The child is unequal to the mere task of the repetition. It can swing its arm three or five times, but would then rest. To do it fifty times without resting is simply too great a task for the child's physical endurance.

Large numbers when used should usually be from real life, newspapers, facts of political, commercial, and industrial relations, etc., where the use of the large numbers is necessitated by the facts of the case.

17. *The premature introduction of difficult matter* whether of processes or of complicated problems.

While not falling into the error of underestimating and starving the child's reasoning impulses, care should be taken not to outrun the powers of the child's undeveloped mind, either by the mistake of forcing it beyond its powers, or, most common error of all, by allowing it to fall into rote work.

What is unduly difficult at eight may be easy at ten.[10] There is a tendency in some quarters to push back the elements of the more difficult topics into the earlier years, overlooking the unnecessary strain thus put on the mind. Germany finds it no drawback to take her boys at the age of nine, with nothing but the four operations on integers. It is a mistake to introduce, for example, calculations with large numbers, or general principles of fractions, or percentage into the work of the fourth grade. Long division, likewise, is easily mastered when the general and arithmetical powers of the child are somewhat developed, but is often made needlessly hard by too early treatment.

Methods

The current methods of teaching arithmetic have been classified as:[11]

Methods of Teaching Arithmetic.

1. Cramming — starting dogmatically from rules.
2. Demonstrative — proving the rule, then working by rule.
3. Heuristic.

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10 "A baby's hand fails to grasp what a well-grown hand can completely cover; there are mathematical ideas usually taught in primary grades which the average mind cannot grasp before the age of fourteen. The average mind learns a mathematical process with extreme quickness when so far developed as easily and quickly to grasp the ideas and principles that are involved in the process, and the time now spent in schools on processes whose principles cannot be comprehended is mostly sheer waste." — Alling-Aber (Mary R.), *Experiment in Education*, New York, 1897.

11 Sonnenschein, E., Vol. VIII., *British Special Reports*.

After what has previously been said, nothing need be added to indicate that from the point of view of the present work, the teaching of arithmetic should be heuristic, though not in a narrow or strict sense, and by no means casting every detail in the heuristic mould.

By a series of easy questions the pupil may usually be led to see the desired relation or process, which is then fixed in mind and mastered by a succession of easy applications. Last of all, the form of description of the process, a rule, may be stated.

The teaching of the multiplication table may be taken as example. In such teaching, no table would be given to the child to be memorized, but he would first be led to find out for himself, by use of objects or some concrete material, what the various multiples of 2 are.

Then these multiples would be fixed in mind by a sufficient number of easy problems utilizing them. These problems should be interesting in themselves, and not simply repetitions of the abstract relations like $2 \times 7 = ?$

Last of all, he may write out himself a table of the multiples of 2 which he already knows. If mechanical "drill" is needed to fix these multiples in mind, it may come now and later.

The multiples of 3 are treated similarly (among the applications, 3 ft. = 1 yd.). In the problems, multiples of 2 may also occur, and so throughout, multiples previously treated should be kept in mind by incidental occurrences in problems.

Finally, when all the multiples have been so treated, the whole may be summarized in a formal multiplication table.

This teaching would not taboo rules, but would have them reached and formulated by the pupil. Under these conditions they are not dogmatic orders, but forms of statement convenient for remembering what the pupil has himself found out.

Such teaching would also occasionally state dogmatically a needed rule that is too hard for heuristic development, and would permit its mechanical use thereafter, though it would require that the questioning powers of the mind be so exercised and developed that the reasons can "flash in" later![12] Still more would it permit the reasoning whereby a mode of procedure had been established to lapse into the subconsciousness, while the procedure continues to be employed. With occasional conscious recurrence to the theory and with the preservation of the heuristic attitude of mind in a general way, such rests contribute largely to a more firm and mature grasp.

The methods named above relate to the manner in which the truths of arithmetic are presented to the child. The order in which the topics are presented has also been made the basis of a method known in America as the "spiral method," in Germany as "*Die Methode der konzentrischen Erweiterung.*" This method proposes to recur to each of the principal topics of arithmetic many times, and at more or less regular intervals. Each recurrence is to strengthen and enlarge somewhat the child's knowledge of the topic.

The Spiral Method.

The method represents a reaction from the treatment of each general topic as a well-rounded whole — to be completely treated in a chapter by itself, before any other topic is taken up. Like all reactions, it sometimes manifests a tendency to go to the other extreme. It does well in bringing into the foreground the fact that the child's attention should not be too long concentrated upon a single topic, nor, on the other hand, should a topic be left too long unreviewed. The importance of reviews, and especially of *incidental* reviews, while ostensibly developing some other topic, cannot be overestimated. On the other hand, even the child has his modicum of continuity, and will be disturbed by too rapid changes.

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[12] "Dr. Stanley Hall holds that a good deal of arithmetic should be taught technically — that is, processes may often be shown first and examples given, the reason for the process being left to 'flash into' the mind at a later stage when reason is more maturely developed." — Barnett, p. 288. No reference to Hall's original statement.

Few educators would any longer claim that large topics, like each of the fundamental operations, fractions, percentage, and the like, should in turn be treated fully and finally at one time: few would go to the other extreme and claim that the topics should each be made the subject of a single lesson in turn. Perhaps it will yet require considerable experience to show what is the real golden mean.

Occasional thorough reviews from the beginning seem well worth trying — perhaps at the commencement of each year. In France, the work of each year begins at the very beginning, simultaneously reviewing and extending what has gone before. So that even the class of *mathématiques speciales* (taking up work which our stronger institutions give as electives in the later collegiate years) begins its work with notation, numeration, and the four fundamental operations; and passes in review the entire field of arithmetic, treated vastly differently of course than in the earlier years.

What has been said in a previous chapter on the subject of definitions applies with especial force in arithmetic. Here,

Definitions in Arithmetic.

above all, formal definitions should not be the basis of the work, but the *things themselves* should be used and the terms correctly exemplified. Formal and precise definitions are as much out of place in the class in arithmetic as elsewhere in the young child's experience. Everywhere the child is acquiring his vocabulary, is gradually learning from usage the more precise demarcation of terms. He gets the idea of *river* from the rivers he sees, not from a formal definition or from a comparative discussion of the terms "river," "stream," "creek," "brook," "estuary." This comes much later. Arithmetic should be no exception to the general rule.

Passing to the definitions of particular terms, only one or two can be taken up here.

Quantity (arithmetical) has been defined as that which can be increased or diminished. Anger or affection can be increased or diminished, yet they are not arithmetical quantities or measured by arithmetical quantities. Those things only are measurable by arithmetical quantities, concerning which we have a clear idea (or definition) of equality. Thus two line segments may be equal, but not two emotions. *What* constitutes the equality depends on the character of the things. To bring things within range of arithmetical treatment it must be possible to suppose them composed of equal parts; these parts are paired off with the series 1, 2, 3, 4, . . . To make an arithmetical study of an emotion, the psychologist must turn to those of its manifestations or effects for which equality can be defined and tested.

In arithmetic, we also suppose equality so defined that things which are equal to the same thing are equal to each other; that is, if $A = B$, and $B = C$, then $A = C$.

This is not necessarily so. In sensations of weight, for example, a weight A of 10 grams produces the same sensations as a weight B of 11 grams; likewise the sensations of weight produced by B and C (12 grams) are the same; yet when A and C are compared, C is felt to be distinctly the heavier. That is, $A = B$, $B = C$, $A < C$.[18]

Multiplication is often defined as "finding a number, called the *product*, by doing to a given number, called the *multiplicand*, what is done to unity to produce a given number, called the *multiplier*." The multiplier 2, for example, is $1 + 1^2$, and consequently, according to the definition 2×4 would be $4 + 4^2$ or 20. The multiplier 2 is also $1 + \frac{1}{1}$; hence 2 times 4 is $4 + \frac{4}{4}$ or 5. The multiplier $\frac{3}{4}$ may be made up from unity thus: $\frac{1}{4} + \frac{1^2}{4} + \frac{1^3}{4}$, and $\frac{3}{4} \times 8$ would be $2 + 16 + 128$ or 146. It is to be noted that these examples presuppose knowledge of multiplication of integers. This does not diminish the force of the illustrations, since the definition under discussion is not usually brought out until multiplication of fractions is reached.

Careful consideration will show that the extension of the term "multiplication" to fractional multipliers is merely a

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[18] Poincaré, *La Science et l'Hypothèse*, p. 35.

definition,[14] though it is naturally made. In practice it is quite sufficient simply to state that a fractional multiplier means that fraction of the multiplicand, and proceed to work problems applying this meaning. For while it is not possible to look out of the window 2½ times, it is possible to turn a crank through 2½ revolutions. As the price of 3 lbs. of butter is 3 *times* that of 1 lb. and that of ¾ lb. is ¾ *of* the price of 1 lb., it is natural to define that ¾ times 32 shall mean ¾ of 32, so that we can say in all cases n lbs. of butter cost n times the price of 1 lb. whether n be integral or fractional.

Rules in Arithmetic. Two extremes with respect to the treatment of rules have been advocated. The first makes the work of arithmetic largely the mechanical application of rules dogmatically stated. The other attempts to banish rules entirely from the subject. The golden mean would seem to lie in leading the child first to understand the process by repeated thinking out of all its steps; second, to notice that in all the problems the procedure has been the same; then to describe the process so that it can be applied in other problems without thinking out the whole process anew. This succinct direction is a *rule*. In other words, rules are very valuable auxiliaries when they are reached by the child himself under guidance and assistance; when he recognizes a rule as a convenient way of stating for permanent preservation and use, the processes which

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[14] Of course a more general definition can be set up covering both integers and fractions, by regarding integers as fractions with denominator unity. Thus, the definition, *The product of two fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators*, covers the possibility of one or both factors being integers. But such considerations are entirely too difficult for beginners. Fractions are usually defined concretely in elementary work as parts of some (geometric or physical) magnitude. No doubt this is the best definition for elementary instruction, in fact the only practicable one, but it is possible to define them as combinations of two integers, and make the partition of magnitudes an application rather than the defining characteristic of fractions. This is done, for example, in Tannery's *Arithmétique*, Paris, 1900.

he has found and used in a tentative manner. With respect to the topic to which it relates, the rule is a *summary*, at the close of the work; for the future, it is a *tool* to be used whenever needed.

Character of Proofs in Arithmetic.

The usage of the best teachers of arithmetic to-day finds the wisest course between the extremes on the one hand of mechanical rules, arbitrarily given and blindly followed, and abstract, general demonstrations on the other. By means of special instances, very simple at first and increasing in difficulty only as the easier ones are clearly understood, the pupil is led to see for himself the procedure that is essentially common to them all and then is ready for the expansion of this procedure in an explicit statement. In many cases he may even be led to formulate this statement himself.

For example, multiplication of decimals might be taught by taking up first a series of problems with integral multiplier until the pupil sees clearly that the number of decimal places is the same in the product as in the multiplicand, that 5 times 1.27 (127 hundredths) is a number of hundredths — just as 5 times 127 chairs is a number of chairs; next, integral multiplicand, and .1 (.2, .3, etc.) as multiplier, then any multiplicand and .1, .2, .3, etc., as multipliers; then the pupil may be led to see that just as we multiply by 57 by multiplying by 50 and by 7 and adding the results, so we can multiply by 5.7 by multiplying by 5 and by .7 and adding the results.

He is now ready to multiply any number by a multiplier which involves tenths, — and has the essence of multiplication of decimals. Having achieved this distinct step in advance, some other topic will be taken up, for the child-mind must not be tired by too long and monotonous drilling on the same subject. Upon later recurrence, what was gained above may be reviewed and can be extended to hundredths in the multiplier without difficulty. Last of all, when the process is quite familiar, the mechanical rule about counting the number of decimal places in multiplier and multiplicand might be formulated if the teacher deems it best to do so.

Very few operations are so simple that they cannot be performed in different ways. Even the fundamental operations are no exceptions. The question then arises: To what extent is it advisable to teach only one form of solution, and to require adherence to this form?

Varying Forms of Solution.

It would seem that a simple form should be carefully taught first; that will of course be used which, on the whole, is the most easily understood and applied. After it has been sufficiently mastered in theory and practice, alternative forms may be introduced if needed, and compared with the form first taught. After one mode of solving some type of problem has been sufficiently mastered, it is instructive and helpful to study other forms also, but it is questionable whether as a rule advantage is gained by teaching a second mode while the first is not comprehended. It is easier not to change horses in the midst of the stream.

The importance of oral arithmetic is generally recognized. The term is here used to cover all work of which none is written, whether it is actually rehearsed in words or simply performed in the thoughts. In this sense, the typical form of arithmetic is oral; writing is an aid that is used when the numbers involved become too large or the relations between them too complicated to be carried accurately in the mind. Written work is indispensable, but it is so only, because of the weakness of the mind.

Oral Arithmetic.

This gives a hint for the order of written and oral work. The oral work precedes; in it are cultivated especially the idea of number, intelligent grasp of the numerical relations and processes, clearness of thought and speech, the power to grasp a problem clearly and to make simple inferences. In it, principles and methods are made clear by use of very small numbers; when these are understood, the numbers are gradually made larger and the data more complex. When no longer easily carried in mind, written work comes to the rescue, and of course has its own technique to be learned.

Written and oral work are complementary phases of the same subject, the same instruction. To separate them into two distinct subjects, with separate class periods, text-books, and topics, is as unwise as it would be to have one class period regularly devoted to oral geography and another quite separately to written geography.

The Subject Matter of Arithmetic

The Number Concept.

The real nature of the notion of number (integral) has engaged the attention of thinkers from the time of the Greeks to the present day. Many views have been advanced as to what constitutes number; some regard it as identical with things, the multiplicity of things; others as a quality of things (somewhat analogous to color) not identical with the things but still not existent apart from them. Others regard it as a mental process (McLellan and Dewey), still others as a symbol ("a locution and a sign," Laisant-Lemoine). These various views are interesting and important, and influence the mode of approach adopted by teachers and writers of text-books. But they play no further part in the direct work of instruction, and discussion of the number concept may be omitted here. The child undoubtedly reaches the idea of number in a very concrete way. Simple objects suitable for counting are within the reach of all, not the least valuable being the ten fingers, and most children have learned to count with small numbers before entering school.

Counting vs. Measuring.

In recent years the view of which Sir Isaac Newton was a prominent exponent has been revived, namely, that all number is ratio and implies measurement. This view can, no doubt, be ably defended on the basis of a suitable definition of the terms used, but as ordinarily understood, there is an important difference between counting and measuring. It is, in mathematical parlance, that between discontinuous and continuous quantity. One *counts* discrete objects by pairing them off, one by one, with the objects of a standard set, namely, the natural numbers, one, two, three, four, etc. To measure, whatever is measured must first be divided (in thought at least) into discrete parts (units of measure), and then these parts are counted. As thus regarded, the act of measuring is more complicated than that of counting, since it implies counting and another operation.[15]

True, one may explain the term "measuring" so as to include counting, but the usage is not in accord with common parlance, and the gain is not apparent — while there is an obvious loss in weakening (by widening) the significance of the term "measuring" and also by running counter to common parlance in the use of so common a term. One would not say that he had measured the number of men in a room with the unit *one man* and found seven, but simply that he had counted seven men in the room. At bottom, the psychologic process is no doubt the same, but there is a real distinction between counting and measuring, as set forth above, which offers no difficulty in instruction and which it seems a pity to abandon.

That the child should early learn to measure, and that measuring and counting should go hand in hand, seems sound. In how far the ideas of comparison, of measurement, of ratio, deservedly prominent in arithmetic from an early stage, should be allowed to *dominate* the methods of instruction, is a question that may properly be settled by experience. The good teacher will be open-minded; he will test impartially whatever gives fair promise of helping to give the child a clearer grasp of number relations, and, proving all things, he will hold fast that which is good.

The four Fundamental Operations.

There are various possibilities as to the order in which the four fundamental operations should be taught. At present, the net result of experience and theory seems to be that these operations should be taken up for successively enlarged fields, as 1–5, 1–10, 1–20,

[15] "To comprehend a ratio requires more than twice as much intellectual effort as to understand a simple term. In three-fourths ($\frac{3}{4}$) we have to think first 3 and then 4, and then their relations, 3 as modified by 4, and 4 as modified by 3 . . . three steps must be retained all together."— Harris, *Psychol. Foundation of Education*, New York, 1898, p. 346.

1–100, 1–1000; considering all four operations in each case before widening the field, running neither to the extreme of treating each operation completely for the whole number field or even a relatively large field before the next operation is taken up, nor on the other hand to that (Grube) of enlarging the field a single number at a time, and continually repeating all the operative combinations for each number.

Addition.

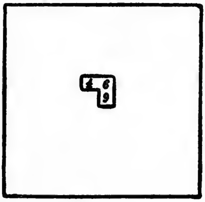
In the teaching of addition the fact must be remembered that the addition of any set of numbers, no matter how many or how large, consists in the repetition of a single step, namely, the addition of a number less than 10 to one less than 100. The *drill*, by which alone accuracy and speed in addition can be attained, may hence be confined to small problems of this character. The mistake is often made of burdening pupils too soon with long problems in addition. Whoever can state accurately and quickly the sum of an integer less than 10 and another less than 100 has mastered the most difficult part of addition.

An excellent form of drill that has stood the test of time is counting by 2's, 3's, . . . 9's, beginning in turn with every number less than 10 and continuing to 100 or a little beyond. More difficult and very helpful would be counting by steps alternately equal, as, 4, 6, 13, 15, 22, 24, 31, 33, 40, 42, etc. When all possible combinations of two steps have been mastered the foundations of accurate and rapid addition have been laid. With the exception of some training in rapid recognition of groups of digits whose sum is 10, it is doubtful whether time can be spent to advantage on devices for abridging the work of addition. The machinery of addition is never committed to writing, hence the drill is necessarily oral. But the eye must be accustomed to see the digit to be added. To this end, a line of digits may be written, as 2, 7, 3, 6, 4, 5, 9, 8, 1, and some other, as 57, added to each in turn.

Tabular arrangements that record many problems in a small space are easily devised. For example, for addition, subtraction, multiplication, and division, arrange a table of numbers on a large card or on the blackboard; for example:

6	8	4	7	2	9	3	1	8	5	0
3	7	8	1	0	2	5	9	9	4	6
5	4	8	2	6	3	7	9	0	1	7
9	5	0	2	1	8	4	3	6	7	5
2	4	6	8	0	1	3	5	7	9	2
1	3	5	7	9	2	4	6	8	0	1
8	1	4	7	0	3	6	9	2	5	3
7	0	3	6	9	2	5	8	1	4	4
4	2	1	3	3	4	1	2	3	2	8

A card large enough to cover the whole table and from which a piece has been cut as indicated may be held in position at various places on the table, selecting thus two numbers for oral addition, subtraction, multiplication, or division, as the case may be. For written addition and subtraction larger openings could be used. Many analogous devices can readily be made by the teacher with but little trouble and ingenuity, and once made, are a permanent source of varied drill problems without further effort.



Subtraction.

To simplify subtraction additive subtraction has been proposed. It consists in regarding a problem in subtraction as asking: "What must be *added* to the subtrahend to produce the minuend?" The advantages of this method are that it applies the "addition tables," dispenses with "subtraction tables," and minimizes the difficulties connected with "borrowing."

The process itself is not shorter than the traditional one, but the following observations and experiments have led me to believe that the mind falls in with it more readily and easily. On asking myself questions like 95–67 = ?, I found that I added sufficient to 67 to make 95. Repeated tests convinced me that I had unconsciously been using additive subtraction in mental work long before I had ever formally heard of it.

In written subtractions of large numbers the method learned in childhood had not been displaced in the workings of my own mind. Inquiry of a considerable number of people, in business and society, led to the discovery that in almost every instance additive subtraction is used, except in those cases where there was a conscious mental revival of the method learned at school. Clerks in shops almost invariably use this method to find the remainder or "change" due the customer. The fact that in so many instances additive subtraction has been adopted unintentionally, seems to indicate that it is decidedly easier for the mind, and might well be taught in the schools.

Division and Partition.

The distinction between division and partition has received far more attention than its importance warrants. No doubt there is a logical distinction between the questions: *How many 4's in 12?* and *If 12 be separated into 4 equal parts, how many units in each part?* And between: *If 5 yards of cloth cost 75 cents, what is the price per yard?* and: *At 15 cents per yard how many yards can be bought for 75 cents?* But this distinction does not affect the mathematical process used, nor need it be mentioned to the child. The essential thing for him is to know *what* the result will be (cents, yards, etc.). He will have no difficulty in seeing this immediately from the given data, if he is not confused by technical terms. The numerical part of the result is determined by the same abstract operation (division) in each case. Similar remarks may be made with respect to multiplication. The concrete nature of the multiplicand, the divisor, the dividend, determine the concrete nature of the result. Its numerical magnitude is determined by an operation with abstract numbers.

Problems can always be stated so as to bring this out clearly. For example: At 15 cents per yard the *number* of yards which can be bought for 75 cents is the *number* of times 15 is contained in 75. If 5 yards cost 75 cents the *number* of cents per yard which it costs is the *number* of times 5 is contained in 75.

Similarly: At $1 per table, what must be paid 7 workmen for making 5 tables each? Here: *Number* (of workmen) *times number* (of tables) *equals number* (of dollars paid).

It is not difficult thus to bring the number idea into the foreground, though in instruction it need not be explicitly done unless required to clear up difficulties. The child will think informally, but correctly. No one thinks of dividing 75 cents by yards, but by 5 (the *number* of yards). And seventy-five one-cent coins are not actually taken and made into five equal heaps. Such concrete manipulation is soon dropped, and not long thereafter even the imaginary making of the heaps is also dropped, and one thinks simply of the numerical problems. We seek *cents* per yard, and the number of them will, according to the data, be the quotient of 75 by 5.

Fractions.

It is no doubt true that the notion of *fraction* is more complex than that of integral number,[16] but at the same time, the idea is so easily made concrete, arises so spontaneously in the child's experiences, that by keeping strictly to the concrete, by following the needs of the child's motor activity, by avoiding all the more abstract questions, the general concept *fraction* and the simplest operations with fractions can be easily mastered even in the first treatment of division. The more systematic treatment might well be deferred perhaps later than is customary, and taken up gradually over a longer period of time, deferring the full theory to the period when the use of letters shall be well in hand.

Various views have been held as to the relation of decimal fractions to other fractions. They may be taught before the latter, simultaneously with them or afterwards.

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[16] "The child finds that it requires a double act of the mind to think quantity at all, for he has to start with quality and then abstract from it, or think it away. But he has to double this mental act again to think a fraction. He thinks the simple number 8, then 7, then he combines them in one thought, and his result is the thought of the fraction, *seven-eighths*." — Harris, *Pyschol. Foundations of Education*, p. 345.

In the first case, they would be taught simply as an extension of our decimal system of notation. In countries using decimal weights and measures, the current measures furnish concrete illustrative material already more or less familiar to the child, and decimals are a natural by-product of the study of denominate numbers.[17]

Decimal Fractions.

In the second case, decimal fractions are treated simply as a different notation for certain types of fractions, the notation having some elements of convenience but involving no new principles, and hence to be taught as a phase of the main subject, fractions.

In the last case, the view is held that the child cannot easily form a clear idea of decimal fractions, whose denominators are necessarily large in comparison with those of ordinary fractions; that his work is consequently purely mechanical, and that all the thought value of the subject is attained from common fractions. From this point of view decimal fractions should be introduced late and briefly after the whole theory of common fractions has been well grasped.[18]

There is a certain amount of truth in each of these positions. Undoubtedly, the child has no clear conception of

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[17] "In practical life, decimal fractions, have become the usual, the 'common' fractions. They are remarkably concrete, easy and simple. Consequently since common fractions have lost nearly all practical and scientific importance the subject should consistently be restricted to a very few exercises; the school will have done its duty if it gives the child a clear understanding of common fractions with exercises in reading and writing them and changing to other denominations." — Condensed from Knilling I., pp. 278–280.

[18] "Although pretty much all real mathematicians have declared themselves against the custom of taking up decimals before common fractions, it seems to be rather increasing than decreasing. It is a deplorable consequence of our decimal system of measures. Though I know it will be in vain, I adduce once more the reasons against this idea:

"1. The fraction $\frac{7}{10}$ assumes the concept of fractional units just as much as $\frac{7}{9}$.

"2. The rules of addition and subtraction can be understood

76,483 millionths. On the other hand, wherever decimal measures are in vogue, he hears of tenths, hundredths, and even thousandths concretely, at the grocery and meat market, and they are so easily made real by measurements in school, that it is practically impossible to avoid the early introduction of decimals.

In England and America, the current weights and measures not only give us no help in the subject of decimals, but provide additional difficulties of their own. In America, however, decimal currency furnishes a basis for the introduction of the decimal notation and the idea of tenths and hundredths. Progress once made to this extent, there is apparently no strong reason against acting on the fact that there is no difference, other than that of notation, between decimal and common fractions, and treating the two notations in a general way side by side.

Short Cuts.

While it is easy to overdo the teaching of devices which in special cases are shorter than the usual process, there are certain ones which are very useful and should be taught (for example multiplying by 25 by annexing two zeros and dividing by 4). The object of teaching them is not only their actual utility in computation, but also, and indeed chiefly, to show the child how to improvise

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only with difficulty, those for multiplication and division not at all.

"3. Decimal fractions are not adapted to oral work.

"4. It is a needless departure from the historic order.

"5. The decimal system of measures is a consequence of our system of decimal fractions and not conversely.

"6. Pupils do not grasp the great importance of decimal fractions and the great simplification which they effect, before they have experienced the difficulty of working with fractions with unlike denominators.

"7. The pupils do not understand why they are burdened with the theory of common fractions afterwards. Logically, the treatment of common fractions should be confined to one operation: To convert a common into a decimal fraction." — Simon, *Math. Unterricht*, p. 47.

short cuts for himself. The power of seeing how to use general results to abridge particular processes is a very desirable one to cultivate.[19] Many lists of "short cuts" found in various books are mainly rules of algebra, mensuration, or geometry, adapted to special cases. A "short cut" is by its nature only useful sometimes, and the ability to see the opportunity to abridge is well worth cultivating.

By teaching a few of the simpler and more useful of such devices before the usual process becomes thoroughly automatic, the use of the short cuts will become automatic along with the use of the full process.

Proportion.

Only the simplest phases of proportion seem needed in arithmetic, and they may well be treated informally and untechnically. The idea of the proportionality of numbers underlies a considerable part of the material used in arithmetic. The child readily sees, for example, that if cloth costs 75 cents per yard, 5 yards costs 5 times 75 cents; 10 yards costs twice as much as 5 yards; and, in general, that if the number of yards be multiplied by a certain number, the price is multiplied by that number. The inverse is likewise clear. If there are $\frac{1}{3}$ as many yards, their cost is $\frac{1}{3}$ as great. These two steps are all that is needed for the solution of the problems in proportion occurring in arithmetic, the method being that of reduction to unity or "unitary analysis."

Proportion need not be made a separate topic. It is simply an application of fractions. All requisite ideas can be developed by a few questions. As a rule, the intermediate steps are merely *indicated*, and the final result computed after cancelling. Throughout, no technical terms are required, in fact, they tend to obscure what is essentially simple.

The square root or the cube root of small numbers that are perfect squares or cubes can be seen by inspection, and thus the meaning of square root and cube root can be made clear.

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[19] See Scripture, Arithmetical Prodigies, *Amer. J. Psychology*, Vol. IV., pp. 42–45.

The general process for the extraction of cube root is now deservedly disappearing from the course of study in arithmetic, though square root is still retained, probably on account of the possibility of geometric application.

Square Root and Cube Root.

One might question whether it would not be better to defer square root to the high school work in algebra and geometry, but so long as it is retained in tne arithmetic of the grades, it would seem a fitting exception to the usual course and might best be taught simply as a mechanical rule.

Various proofs may be found in current texts on arithmetic; they seem simple to adults and would relate themselves easily and clearly to phases of geometry and algebra, but it is doubtful whether children as a rule make much out of them in arithmetic. In fact it is almost impossible to help a child to master either a geometric or an algebraic proof. The subject is essentially too difficult and abstract. The result should always be verified by squaring.

It is almost a truism to say that, whether the point of view be that of science, of business, or of teaching, the metric system of weights and measures is decidedly preferable to that still in vogue in this country and the British Empire.

The Metric System.

The opinion of the scientific world may be clearly read in the fact of the general use of the metric system by scientists of all countries.

From the point of view of teaching the use of the metric system would be a distinct gain. The mathematics of the metric system being precisely that of decimal fractions, it can be taught with these, and not only requires no additional time, but actually saves time; for the ease and efficiency of the teaching of decimals is increased when decimal weights and measures are in every-day use about the children. They furnish the concrete basis for the teaching of decimals as well as its utilitarian justification and necessity. We may really regard as wasted all the time and energy expended in the teaching and learning of our weights and measures.

The objection of a French writer [20] that the very ease of the metric system means a loss of discipline and consequently retarded growth of mathematical thought power would be well taken if there were no other mathematical subjects, of equal disciplinary value, on which the time saved could be profitably spent, but when choice is embarrassed by the very wealth of mathematical material and its wide range of applications that are within the grasp of the children and of interest and later value to them, there seems little need to retain for disciplinary reasons what should otherwise be discarded.

From the business point of view, the greater simplicity and ease of calculation with the metric measurements commends it highly. It would be difficult to estimate the amount of time that would be saved if all calculations involving weights and measures were made on a decimal basis. One may readily test for himself by computing the volume in cu. ft. of a rectangular tank 4 ft. 8½ in. by 3 ft. 11¾ in. by 2 ft. 5¼ in., and the volume of a similar tank 1.32 by 1.07 by .65 meters. As a still further test one may find how many gallons the first will hold and how many liters the second.

At the time of adopting the metric system there would doubtless be some difficulty in effecting the change of systems, but this difficulty will grow greater with each year that the change is deferred. The metric system is now thoroughly established in so large a number of countries that it is in fact *the* international system. Of the leading civilized nations, only the British Empire and the United States are not yet using it.[21] These nations will not permanently consort with China in refusal to introduce a manifest improvement, and the sooner the change is made the more easily will it be made.

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[20] Leysenne, *Traité d'Arithmétique*, 4th ed., Paris, 1897, p. iv.

[21] See report of the hearings before the Committee on Coinage, Weights, and Measures of the 57th Congress (1902), especially pp. 40–46.

The advantages of the metric system are conceded on all hands, by business men and manufacturers, by teachers and scientists. We now simply await the time when our legislators will be willing to introduce this simple and rational system in lieu of our irrational system now nearly antiquated in the civilized world.

Teaching the Metric System.

It is certainly desirable that all teachers should appreciate fully the great superiority of the metric system and should join, as opportunity offers, in the agitation for its adoption. But the desirability of teaching it systematically in the grade schools, with actual measurements and much calculation, — in short, as though it were the system in vogue in our country, — is not so evident. For the children in these schools, the metric system lies entirely outside the field of experience. It *is* a foreign system at present, much as we may wish that it may not long remain so. The school will note a marked gain of time when it is no longer necessary to drive our irrational system into children's heads; while this must be taught, it is a loss of energy to teach also a foreign system, however good in itself.

As a part of the propaganda for the introduction of the international system, it is surely useless and out of place to teach it in the schools. What is needed is to influence Congress, not to burden little children with a foreign system of weights and measures of which they see or hear nothing except in the school-room.

If the metric system were difficult to learn, if it were taught at a disadvantage when needed, or if it had any disciplinary value peculiar to itself, these would be strong reasons for teaching it to all young children. But as its very merit is its simplicity, there seems to be no need to teach it before the occasion arises for its use. For the ordinary child, no such occasion arises before the science work of the high school. All that is then requisite for him to learn is a few *names;* the mathematical theory and practice he has already had in connection with decimals.

The Algebraic Side of Arithmetic

The use of letters to represent numbers is within reach of the child at an early stage and can be understood easily and naturally. It simplifies the treatment of some types of problems which are otherwise prolix and tend to confuse by mere verbiage. The child is accustomed to use J. for John or Jane, N. Y. for New York, and has no difficulty in using similar abbreviations in arithmetic as soon as he has occasion to make general statements. That A. may to-day stand for area, to-morrow for altitude, will trouble him no more than that J. may stand for Jane as well as for John. When he can formulate the statement that *the number of square units in the area of a rectangle is the product of the number of linear units in its length and its breadth*, he is also ready to understand the abbreviated statement: $A = L B$. These abbreviations come in so naturally that all that is needed is to abandon the determination to keep them out.[22] Their use in simpler equations will cause no difficulty but will simplify many a hard problem.[23]

The Use of Letters to represent Numbers.

The extent to which letters can be used in the solution of problems will dictate the extent to which drill upon manipulation of literal expressions should be carried. There should be no attempt at completeness, and no introduction of pro-

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[22] "Every child is a natural symbolist; a corncob with a dress on it will do for a baby, and a stick with no additions for a horse. To let one thing stand for another is as easy to a child as to breathe. Advantage of this can be taken to teach comprehensive formulae, $a \times b = c$ should be the child's general expression for addition from the first primary year." Alling-Aber, *Experiment in Education*, p. 171.

[23] "The pupil must see that calculation with letters is in most cases simpler than with numbers, that the laws according to which we calculate are the same in both cases: and that the difference lies only in the fact that the value of the letter is not known." — Bardey, Preface to *Arithmetische Aufgaben*.

cesses not actually needed. The work should remain an integral part of arithmetic, its literal phase.[24]

The use of the equation is coextensive with arithmetic, beginning with $1 + 1 = ?$, or what is the same thing, $1 + 1 = x$.

The Equation. The expression "use of the equation" is, however, often applied particularly to those cases in which the equation must be manipulated more or less in order to find the value of the unknown quantity. Thus the equation:

$$5n + 17 = 52$$

(formulating the problem: What is the number whose five fold increased by 17 makes 52?) must be subjected to several transformations before the direct computation of the number can be undertaken, while in the equation:

$$\frac{52 - 17}{5} = ?$$

the unknown number can at once be computed.

There is, however, no radical distinction between the one form of equation and the other. The complexity of the equation grows with the complexity of the problem. The use of letters facilitates the advantageous use of the equational notation, — and the transformations needed are easily justified by appeal to the common sense of the child, and the idea of *balance* between the two numbers which are equal.

The formal study of equations for their own sake, their transformations and solution, falls within the provisions of

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[24] "There is no reason whatever why all study of algebra should be postponed until the whole of arithmetic has been mastered. Of course it cannot be begun until the scholar has acquired a certain power of abstract reasoning.

"The solution of simple equations and of problems which lead to them may also be done at a very early stage. The advantage of this is that the pupil's interest in the applications of algebra is aroused and he is compelled to work in an intelligent way. The problems cannot be too easy at first and they should be carefully graduated." — Mathews in Spencer, *Aims and Practice of Teaching*, p. 182.

algebra. In arithmetic, and at the age at which a child usually studies arithmetic, any formal study of the more difficult phases of equations would be premature. In arithmetic the equation is always a tool, and the extent to which it can be profitably used is marked out by the problems of arithmetic itself. Its very simplest phases suffice to make it a remarkably useful tool in arithmetic.

Technic of Literal Arithmetic.

The use of letters to represent numbers opens the door for the application to literal numbers of all the fundamental operations of arithmetic, as well as of the combinations of these operations, fractions, factoring, and the like. A certain degree of mechanical facility in the manipulation of literal expressions must be acquired by practice, and if the simpler types and forms only are taken up, the beginning of this practice may well be made in the eighth grade. Children enjoy this variety of calculation, but care must be taken to keep the problems simple.[25]

Negative Numbers not needed.

Negative numbers may well be deferred to a later stage. They are not needed for the work in arithmetic, and their introduction may mark, in a sense, the transition from literal arithmetic to algebra; from the work of the grades to that of the secondary school.

There are two difficulties in the beginning of what is ordinarily called algebra, the first, the use of letters to represent numbers, the second, the generalization of the number concept from absolute to relative numbers, positive and negative. These two points are quite independent, and there is no reason why the difficulties should not be separated. The first and easier presents itself naturally in connection with arithmetic, and leads to literal arithmetic, the letters representing absolute numbers. When this idea has become quite familiar by use the extension of the number concept may be undertaken.

It has been urged that the use of literal and equational methods in arithmetic makes the work too easy. The reply

[25] See *Report Com. Coll. Entrance Requirements*, p. 21.

to this is that mathematics has enough real difficulties to obviate any need of attempting to *make* things difficult. Nothing can be too easy, if correct. Learn things in the easiest way, and then pass on to something else.

An Objection considered.

The Geometric Side of Arithmetic

The opinion is now widely held that some phases of geometry should receive attention before the high school, and has been specifically recommended by various committees.[26] The purposes of such study are varied; the utility of the knowledge attained, acquaintance with geometric forms and their properties, and the exercise of the powers of observation and intuition.[27]

Geometry before the High School.

The subject has usually been treated as quite distinct from arithmetic, under such titles as "concrete," "inventional," or "observational" geometry. Better results would doubtless be achieved by regarding the subject as an aspect of arithmetic, namely, the application of arithmetic to geometric forms, and *vice versa*.

How to be treated.

Such work would be coextensive with arithmetic. The child lives in the midst of geometric forms, which furnish a supply of simple and easily available material. Good teachers indeed use these forms freely and well in the kindergarten and the earlier school years, but then they are unfortunately often dropped to be taken up, if at all, in the last year under the title "Mensuration."[28] The work should

Extent of the Work.

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[26] See *Reports of Committee of Ten, Committee on College Entrance Requirements*.

[27] "In the form of what the Germans call *Raumlehre*, many geometrical facts would be taught from the first in the proposed curriculum, under the head of drawing and constructive work. When the formal proofs of geometry are later entered upon, they will therefore be seen to be easy and natural rather than difficult and wholly strange." — Butler, *Meaning of Education*, New York, 1898.

[28] "The material and methods of the mathematics should be enriched and vitalized. In particular, the grade teachers must

rather be continued throughout the entire course, based on the measurements and observations, constructions and drawings of the pupils themselves.[29] It must always be of such a character as to carry conviction to the mind of the pupils, which is of the essence of proof, and should of course lead to correct results. This can be done by measurements, by special instances, leading up gradually and informally to truly demonstrative reasoning.

Such a course can be pursued, for example, in those problems which are really only generalizations of concrete cases; such as the proof that the number of square units in the area of a rectangle equals the product of the number of units in its length and breadth (commensurable). Proofs by superposition are also available, as the equality of two triangles having an angle of each and the including sides respectively equal. If the child makes the actual superposition in a few cases, he will readily pass to the general cases, always provided that he is not worried with the requirement to state his reasons with all the "hences" and "therefores."

What Proofs are available.

Quite a little really strict demonstration (as strictness is ordinarily accepted in the schools) can thus be done before the high school, and progress will be still more marked if no attempt at systematic treatment of a list of propositions is made, but only such things are taken up as relate to the work in arithmetic or to practical life, and are easy of ex-

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make wiser use of the foundations furnished by the kindergarten. The drawing and the paper folding must lead on directly to systematic study of intuitional geometry, including the construction of models and the elements of mechanical drawing, with simple exercises in geometrical reasoning. The geometry must be closely connected with the numerical and literal arithmetic." — Moore, *Presidential Address*.

[29] "The first application of arithmetic to geometry should not be of this kind: 'The radius of a globe is 6 in.; find its surface and volume'; but, 'Here on table lies a globe, come and measure its surface and volume.'" — Pickel, *Geom. d. Volksschule* (8te Auflage).

perimental verification, the latter often directly pointing out the demonstrative proof. No little demonstration is successfully done in the Prussian schools before the age of fourteen, but under our conditions it is prudent to make haste slowly. Progress is desirable, but neither the pupil nor the teacher should be forced too fast. A little at a time is here a good motto. Care should be taken to treat the literal arithmetic and the concrete geometry (mensuration) as parts of the arithmetical work. By doing so the interest in algebra and geometry proper will not be dulled by the anticipation of their most striking results in arithmetic.

The Algebraic and Geometric Work, Integral Parts of Arithmetic.

Summarizing what has been said, it seems desirable that the study of geometric form be carried throughout the work in arithmetic with increasing thoroughness as the years pass, that the elements of literal arithmetic be introduced in such measure as they can be utilized, the whole to be one coherent subject, — *arithmetic.*

Perhaps one reason why the attempts heretofore made to introduce such work have not been more unqualifiedly successful, is that in the grammar grades geometry and algebra have been taught as subjects unrelated to arithmetic or to each other; separate texts have been used, separate months have been allotted. The algebra has often been simply an abridgment of the first year's work in algebra in the high school, taking up much of the theory (including negative numbers) but with less complicated exercises. The experiment of considering this work from the standpoint of the schools in which it is taught, as phases of arithmetic rather than the first instalment of subjects to be completed in the secondary school, certainly deserves thorough test.

An Objection considered.

It may be objected that through early and free use of geometric results attained through measurement and reduction the pupil may thus learn to accept insufficient proofs, and that his scientific spirit and appreciation of rigorous proofs may be dulled. The reply may be made: Strict rigor, in the formal sense, is impossible. Every treatise or treatment admissible in the secondary schools rests on a large body of tacit assumptions, axioms. (See the chapter on Axioms.) It is a mistake to reject demonstrations obtained by measurement or by induction from the category of proofs: they are simply based on a large body of axioms. As long as the pupil is satisfied of the correctness of the result, and the result *is* correct, all is going well from the teacher's point of view, and there is no need to stop for anything more rigorous. But when the pupil becomes convinced of the truth of what is untrue, the teacher's greatest opportunity has come. The teacher objects, and perhaps shows by a special case that the result is not always true. If the pupil is convinced, teacher and pupil together hunt up the reason why the pupil was led to believe what is not true, and his idea of what is needed to make a real proof is made markedly more precise. Perhaps, and this is better yet, the teacher is not at once able to convince the pupil; he still believes in the truth of his result, and defends it against his teacher. This defence of his own statements, this searching for the weak points he supposes to be in the teacher's argument, will do more to give him a clear notion of what it really is to prove, than weeks of learning ready-made proofs. But let the teacher beware of overawing the pupil by his authority or prestige. The longer and the more freely the pupil contends (days or weeks, perhaps), the better it will be for him in the end; but if he gives up without conviction, the result is *fatal.*

Miscellaneous Points

Special Points of Method.

There are many other points of subject matter and of method, which deserve detailed consideration. We touch, however, in closing, upon only a few special points of method.

False Accuracy.

The carrying of operations further than is requisite or warranted by the data is a favorite diversion of pupils, and should be invariably discouraged, not arbitrarily, but by giving the reasons. If a measurement is made to the nearest eighth of an inch, it is meaningless to work out one-fifth of the length to the third decimal place. Such calculations lead the pupil to believe that he has determined his result with great precision, when just that part of his result on which he plumes himself as constituting its precision is altogether untrustworthy.

Estimates.

The making of rough estimates of what the result will be should be encouraged, and practice in it given, as well as in observation of whatever may be seen as to the character of the approximation, whether the correct result is larger or smaller, etc. Thus: 488×27 is certainly less than 15,000 (500×30) and more than 12,200 (488×25). Such rough, but not random, estimates serve as a first check, and would obviate some ridiculous blunders.

Checks.

Whenever it is possible, the work should be checked in some way. This is usually possible; if not, a second working, independent of the first, is at least always possible. It may be said, "Train the child to absolute correctness the *first* time." Impossible! all men are fallible. The child feels strong need of a check, and will have it. The only question is which: the book of answers, the dictum of teacher, or his own verification. Let him verify, even if he works only one-third as many problems. Each verification is a problem. When by repeated verifications he himself feels that he can dispense with verification, and still *guarantee* the correctness of his results, let him do so. The essential thing is that he get the result right and know that it is right. He will in addition have gained what is more valuable than skill in computation, — well grounded self-confidence. In actual life there are neither answer books nor kind teachers. The work must be *right;* mistakes usually mean loss to those who make them.

Various Checks.

Addition may be checked by adding up and down, subtraction by adding the difference to the subtrahend, multiplication by breaking the multiplier up into two factors and multiplying by each in turn, division by multiplying quotient and divisor, etc. The solution of every problem, however complex, is made up simply of a succession of these operations. Each one of these should be checked before going on. In many instances, especially when the numbers are small, repetition or careful scrutiny of the work of that step is sufficient check. A rough estimate is a useful first check. Particular types of problems sometimes have convenient checks of their own; for example, in problems involving equations, substituting the result found in the relations given in the problem; in square root, squaring the result, etc.

Memory in Arithmetic.

The remarks already made on memory in mathematics in general apply also to arithmetic. When results have first been derived by the pupil, and used until remembered to quite a little extent, they may finally be memorized once for all to save future trouble. Probably the material thus to be memorized is greater in arithmetic than in any other subject.

Labelling Steps.

An excellent habit for pupils to form is that of labelling distinctly the intermediate results. For example: The schedule of a train from Fort Madison, Iowa, to Chicago, taken as 240 miles distant, requires an average speed of 30 miles per hour. On a certain day the train left Fort Madison 2 hours late, and for $\frac{3}{4}$ of the way the engineer ran his train at 40 miles per hour, the remainder at 30 miles per hour. How much was the train late in reaching Chicago?

No. of hours for transit when on time	$240 \div 30 = 8.$
No. of miles run at 40 miles per hour	$\frac{3}{4}$ of $240 = 180.$
No. of hours required for this part of the journey	$\frac{180}{40} = 4\frac{1}{2}.$
No. of miles run at 30 miles per hour	$\frac{1}{4}$ of $240 = 60.$
No. of hours required to make this part of the journey	$\frac{60}{30} = 2.$
Total time for the journey	$4\frac{1}{2} + 2 = 6\frac{1}{2}.$
No. of hours made up	$8 - 6\frac{1}{2} = 1\frac{1}{2}.$
No. of hours still late	$2 - 1\frac{1}{2} = \frac{1}{2}.$

Indicating Work.

In written work, much emphasis is needed on the fact that it is usually better first to *indicate* the whole work before making the actual computations. This will often permit simplifications by cancellation which will much abridge the numerical work. For example: What is the weight of a cu. ft. of bark if a box 6 × 4 × 11 in. holds 2 lbs. 1 oz.?

One would not say: "The box contains 6 × 4 × 11 cu. in. or 264 cu. in. A cu. ft. contains 1728 cu. in., hence a cu. ft. contains $6\frac{144}{264}$ or $6\frac{6}{11}$ times as much as the box, and $6\frac{6}{11}$ times 2 lbs. 1 oz. is 13 lbs. 8 oz.," but rather indicate the work thus:

$$\frac{33}{16} \times \frac{1}{6 \times 4 \times 11} \times 12 \times 12 \times 12,$$

which by cancellation reduces readily to $\frac{27}{2}$.

In oral work, on the other hand, it is necessary to work out each intermediate result, since the mind cannot carry the intermediate steps unless reduced to a single number.

Very Small Numbers in New Topics.

In developing new topics, the numbers involved should at first be kept so small as to prevent the mechanical burden of computation from diverting attention from the new ideas to be grasped. The difficulty in a new topic lies in understanding and using the specific combination of the elementary operations which effects a desired end. These combinations are just as completely present when small numbers are used as when large and unusual numbers are used. The term *per cent*, for example, can be made familiar much more easily and effectually by problems like: "Find 50 per cent of 80," than by: "Find $17\frac{3}{4}$ per cent of $682\frac{7}{8}$." The teacher will constantly find that pupils who stand quite at a loss before some problem will solve with ease another exactly like it, except that the numerical data are simpler and that two or three such problems given with gradually more complex numbers will lead him without any other assistance to solve with ease the problem before which he previously stood helpless. In new topics it is extremely desirable to keep the numerical data so small that the computations can be performed orally, indeed almost automatically, leaving the child free to focus his whole attention upon the essentials of the new problem. When these are understood, larger numbers can readily be handled if needed.

An important phase of arithmetical work is the training of pupils to see problems where they are not directly formulated. In actual life, in business, one must usually make the problem as well as solve it.[30] The data only are known. Some drill in making problems based on given data would seem therefore to be quite desirable in the work in arithmetic and also in later mathematics.

Pupils make Problems.

That a good text properly used facilitates the work of the teacher is generally believed.

Some characteristics of a good text in arithmetic have been well formulated by Baldwin:[31]

The Text.

1. *It will not be a large book.* It will include essentials presented in good form, but will omit rubbish.

2. *It will be modern.*

3. *It will combine oral and written arithmetic.* It will displace the old mental and written arithmetics that still linger in some of our schools.

4. *The equation will be used,* even to some extent in the sixth grade. In the seventh and eighth grades the work in arithmetic and introductory algebra will go on together. This feature will have great educational significance.

5. *Applied arithmetic will have its place,* including metric geometry and applications to physics.

6. *It will, above all, foster thinking.*

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[30] "The man of affairs meets with but few problems such as the arithmetic offers — that is to say, in the stated form in which the pupil finds them in the text-book, but he is constantly confronted with conditions which demand on his part an ability to apply such principles and rules of arithmetic as will fit the case. The proper use of the text-book is to supply some of the material for home work, and to save the teacher of large classes from the danger of being swamped in preparing and solving and correcting problems." — Doggett, W. E., *N. E. A. Proceedings*, 1900, p. 555.

[31] Abridged from *School Management*, p. 350.

The text should from the pupil's point of view be chiefly a collection of exercises. It must have its theory, either underlying the exercises or directly expressed, but this should percolate to the pupils through the teacher rather than be obtained by them from the book in the first place.[82]

The Teacher

The qualifications of a good teacher of arithmetic have been stated thus by Safford:[83]

Qualifications of a Good Teacher of Arithmetic.

1. Quickness in mental operations.
2. Correctness in calculation.
3. Power rapidly to make new problems.
4. Knowledge of algebra and geometry.
5. Ability to teach objectively and to find illustrations.
6. Patience with slow pupils.
7. Thoroughness everywhere.

The Teacher's General Preparation.

The teacher of arithmetic should have a good knowledge of the subject itself, including its wider and more theoretic aspects, good courses in high-school algebra and geometry, in the elements of trigonometry with applications, and in physics, together with some theoretic and practical study of the pedagogy of the subject. The more this minimum can be enlarged along the same lines the better, but in view of the fact that the teachers are not specialists, but have to prepare to teach other subjects also, more than the minimum named may not be practicable, but the minimum itself seems entirely feasible for good normal schools.

It would be a mistake to suppose that any degree of pro-

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[82] "In this study [arithmetic], more than in any other, it seems desirable that the text-book should serve very strictly to supplement the teacher, not to supplant him. Learners should be shown a process, and with the help of the teacher should formulate the reasons, and then, but not till then, state a rule." — Barnett, *Common Sense in Education*, London, 1899, p. 223.

[83] *Mathematical Teaching*, p. 24.

ficiency in the operations of arithmetic themselves, or any amount of previous experience in teaching the subject, can take the place of specific daily preparation for the class-room work. This includes the outlining of a specific plan for the work, the theory to be developed, the oral and written problems to be taken up, and the assignment for the next time.

The Teacher's Daily Preparation.

The consideration of the topic and problems to be taken up should include a sufficiently detailed analysis of the fundamental operations involved to prepare the teacher to make easy problems offhand. In the oral development at least the class exercises will be more free and effective if the teacher seems to give the problems spontaneously, without direct reference to the text. Here that teacher is unduly hampered who treats the text as an inflexible standard to be followed literally in every instance, rather than as a model which indicates the path to be followed, but leaves the number of exercises of each type which are to be used to the discretion of the teacher.

For example, if the topic is least common denominator, the teacher will have thought of the more important denominators and divided them into two or more classes according to difficulty, as 2, 3, 4, 6, 8, and 5, 10, 12, 16, 20, 24, and will give first a sufficient number of very simple problems of the first class, as, "Reduce to common denominator $\frac{1}{2}$ and $\frac{1}{4}$; $\frac{2}{3}$ and $\frac{5}{6}$; $\frac{3}{4}$ and $\frac{1}{8}$," to secure a clear understanding of the process and considerable skill in its execution, and then gradually introduce denominators of the second class and harder problems.

The teacher will also constantly be on the alert for opportunities to enliven the work with problems of local interest, relative to the local industries, population, government, and public works, the topography of the region, and the like. To this end he will seek occasion to acquaint himself with numerical facts relative to such topics, and will find no difficulty in amassing a large fund of material. Good text-books suggest types of such problems, but

Adding Local Interest.

these are inevitably of a general character — the teacher must add the local color. The problems will seem more real and interesting if they relate, for example, to a well-known local factory, to its employees, its consumption of raw materials, its products, their transportation to market and sale, than if they relate to a supposititious factory or to conditions in some other city. Each community has its own peculiar interests, and it is not difficult to make many problems couched in general terms in printed texts, more local in character; when the text speaks of a monument, a gas tank, a post office, a bridge, it is an easy matter to substitute analogous data relative to *our* monument, our gas tank, our post office, our bridge. The pupils' interest will be still further enhanced if they themselves are enlisted in securing the data needed. Suggestions have been made earlier in this chapter as to sources of such data. It is an important part of the teacher's preparation for the class-room to plan for this work.

Improvement needed, and coming.

That the results attained in arithmetic are not commensurate with the time and energy expended, admits of little doubt. It admits of still less doubt that the world of teachers of arithmetic is awake to the gravity of the situation and that remarkable improvement has taken place in the last few decades, second in importance, let us hope, only to that which will take place in the next decades.

CHAPTER XIII

THE TEACHING OF GEOMETRY

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Special Aim of the Teaching of Geometry.

GEOMETRY, perhaps more than any other subject of secondary school mathematics, offers opportunity for attaining all the ends of the teaching of mathematics, and hence there is less occasion to regard any one of them as specially the goal of geometry. It gives ample occasion for exact reasoning, for real induction applied to very simple data, for correlation with other work, with drawing, geography, and the physical sciences as well as with algebra, for exercise of the space intuition, for practical applications, for drill in numerical computation, for training to habits of neatness and exactitude, and for the cultivation of the powers of precise thought and accurate expression. If from among all of these a selection were to be made, development of the space intuition and training to active and careful reasoning might be regarded as pre-eminently the function of the teaching of geometry. This does not mean that the pupil need work out independently all the propositions of geometry, but it does mean that he should work some, and that with this in view he should be led to see the reasons which might lead to the discovery of those theorems whose proofs are given him.

The Mode of Instruction.

In geometry, as in other subjects, no one mode of instruction can be recommended to all teachers alike, or to the same teacher at all times. The teacher who is ready to use this or that method according to the needs of the hour will probably achieve the best results, but it is better to have a hobby and ride it hard than to have no enthusiasm. The essential thing is that the pupil learn to demonstrate by demonstrating. That he should be guided by the teacher and aided by the text goes without saying, — he should not be expected to rediscover the brilliant results of expert mathematicians by his own unaided strength, but with the matter duly simplified, and cut up into small portions for him, he must make real discoveries himself. The portions cannot be too small, the steps too easy, at first. For it is not sufficient to demand that he discover; he must actually succeed in discovering. The elation of success is one of the advantages to be attained. The instruction may be dominated by this idea without requiring the use of any specially prepared text, provided only that the "original exercises" be not regarded as "riders" but as the main thing, and that the demonstrations which may be given in the text be regarded as preparatory, as models, as leading up to the central work. Perhaps even the demonstrations of the text may often be taken up with the class in the heuristic spirit before the pupil is referred to the printed page. After the pupil has some understanding of the character of geometrical proofs, he may frequently or usually try propositions of the text first himself for a reasonable time, referring to the text later for corroboration or for assistance.

Propositions demonstrated should not merely be understood as first given, but looked at from various points of view, used immediately in the solution of problems, when possible both theoretic and applied, until the pupil is master of their essence rather than simply of the form of statement used in the text.

The Analytic Method in Geometry.

The class-room in geometry is the place *par excellence* for the analytic method (see Chapter III.). The adherence to the synthetic form of Euclid has brought the subject certain reproaches well deserved by the method of presentation.[1] These reproaches

[1] Herbart (I., p. 139, Edn. Williams) objects to the tricky proof in which the end is not evident. Suddenly the trap is sprung, and the "mouse-trap proof" is completed.

Hegel (cited by Herbart) also objects to this arbitrary character of Euclidean proof and constructions, since the dominating purpose is not expressed.

Schopenhauer (*Welt als Wille u. Vorstellung*) calls Euclidean proofs "stilted and tricky," "jugglers' pranks," "the truth usually comes in through the back-door."

may be avoided by approaching geometric proofs in the spirit of the discoverer.

If the pupil is to be more than a passive learner, he must be shown the chain of reasoning by which the proofs given in the text might naturally have been discovered. Unless he catches the spirit of geometric analysis, he will never succeed in finding proofs himself. The form of statement in texts seldom makes the genesis of the proofs evident; they must be explained to the pupil, and, as a rule, the treatment of a proposition should not be regarded as completed until the pupil has a good insight into the analysis of the proof.

As example, the following proposition may be taken:

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane, P, of these lines.

Let *A B* and *A C* be the two given lines, and let *A D* be perpendicular to each of them.

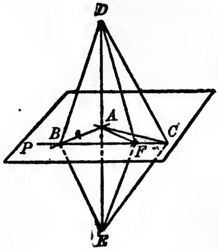
Let *A F* be any other line of the plane *P* through the point *A*.

Draw a straight line in plane *P*, cutting the three straight lines at *B*, *F*, *C*, respectively.

Draw *D B*, *D F*, *D C*.

Produce *D A* to *E*, making *A E* = *D A*.

Draw *E B*, *E F*, *E C*.



Since *A B* is a perpendicular bisector of *D E*, *B D* = *B E*.

Similarly *C D* = *C E*, and hence triangles *D B C* and *B C E* are identically equal.

If these two triangles were superposed, *D F* would coincide with *F E*, and is therefore equal to *F E*.

Since *F* is equidistant from *D* and *E*, the line joining it to *A*, the midpoint of *D E*, must be a perpendicular bisector of *D E*.

Therefore *D A F* is a right angle, and consequently *D A* is perpendicular to *A F* and therefore to the plane *P*, since *A F* is *any* line of the plane *P* through the point *A*.

The form of proof just given is highly satisfactory if the aim is merely to lead the reader along through a flawless chain of steps, every one of which is easily seen to be correct, reaching finally the theorem proposed. Not the slightest objection can be raised to the chain of reasoning as establishing that the theorem is true. For a treatise, a reference-book and repository of facts, this form may be the best and most elegant; it may even be preferable to any other for the pages of a text-book, though this is open to question, but it is certainly not the best form for the class-room. It sheds no light on the reasons for the steps. The dazed beginner is led along, admitting the truth of each statement as arbitrarily given, but seeing no relation between them and the objective statement; he therefore finds it difficult subsequently to hold them all together, and when finally he is unexpectedly brought face to face with the main proposition, it comes to him as a shock. He feels that he is securely trapped, that he cannot get away from admitting the truth of the proposition; but all seems arbitrary and mysterious to him, and he has not the slightest idea of how to set to work to discover such a proof for himself.

On the other hand the following form leads the pupil to see a reason for taking each step before it is taken:

Using the figure and notations above, the proposition will be proved, if we can show that $D\ A\ F$ is a right angle. The facts available are that $D\ A\ B$ and $D\ A\ C$ are right angles.

There are various ways of proving whether or not a given angle is a right angle. Some of these ways may be named and examined as to applicability in this figure, with the result that the most promising seems to be to attempt to prove that $A\ F$ is the perpendicular bisector of a segment of the line $D\ A$, having A as its midpoint.

Geometric Analysis.

Hence lay off $A\ E = A\ D$. We now wish to prove that $F\ D = F\ E$.

How are two line segments proved equal? One way is by superposition.

But we cannot hope to prove them superposable without considering our known facts. One of these is that $B\ A\ D$ is a right angle, and that hence $B\ D = B\ E$.

Similarly $D\ C = C\ E$.

Hence triangles $B\ D\ C$ and $B\ C\ E$ are identically equal and can be superposed.

Doing so by rotating triangles $B\ D\ C$ about $B\ C$, we see that point F remains fixed, and point D is brought to coincidence with E, and hence $D\ F$ coincides with $F\ E$. Q. E. D.

If, as a rule, proofs are handled in some such way, raising the questions and discussing the various possibilities step by step in the class, the pupil will soon be ready and eager to attack propositions himself. For such geometric analysis, demonstrations which are simple and straightforward will uniformly be preferred even when they are longer and less elegant than some artificial demonstration.

The Transition from Concrete to Demonstrative Geometry.

Concrete geometry, in one form or another, may be a constant part of the work in arithmetic from the earliest years. The transition to strict demonstration should not be abrupt, but gradual. Quite a little of informal, but real, demonstration may be done in the last year or two before the secondary school, so that when the pupil nominally begins the study of demonstrative geometry in the secondary school he has already not only a large stock of geometric names and facts, but the spirit of demonstration well awakened. But much growth can still be made through concrete or constructive geometry. It would be a mistake to abandon this side, when demonstrative geometry is taken up as a separate subject. Even when the course of formal demonstration is in full swing, the concrete instrumentalities may still be used to lead up to propositions, to illustrate them, to study their bearings, properties and consequences. The use of models is of great value. They should be made by the pupil himself when feasible, and besides their constant use in proofs they may sometimes be made means of direct proof, for example, in propositions, dependent on dissection or superposition. As another example,[2] let one end of a narrow strip of paper (say 4×18 in.) be given a half turn and pasted to the other, and then the ring thus formed be

[2] Brandford, *l. c.* in *Bibliography*.

cut lengthwise to width 2 in.; few pupils would be able to see clearly by geometric imagination what will result, whereas they would obtain a perfectly rigorous proof of what would result by doing it. A second cutting, to width 1 in., and a third to width ½ in., becomes still more difficult to follow in imagination.

The question may be asked: When should more formal geometry be begun? Whenever the pupil feels the need of it.

When begin Demonstrative Geometry?

Use intuition freely and prove what does not seem evident. The pupil may accept intuition as proof: that is not bad. Many great mathematicians have done so with happy results for the science. As long as he accepts intuitionally statements which are true, *go on* — things are progressing nicely. When his intuition fathers a falsity, *prove* to him that it is false (by concrete example or the like). This will arouse him to a sharp criticism of what constitutes a proof, *your* proof, and will do more to teach him the nature of a proof, than dozens of routine demonstrations learned by rote, or even passively understood. But let the teacher be sure that he convince the pupil fairly, and in no wise awes him into silence by superior knowledge and position.

Euclid as a Text and Model.

On the continent of Europe, the form and character of proofs and the order of sequences adopted by Euclid have long since been materially modified, but in America, and especially in England, the great Greek encyclopedia of geometry has literally or virtually been used as text in much of the instruction of young boys and girls up to the present day. The movements which have been discussed in the chapter on the laboratory method have brought to a focus, however, a strong tendency away from Euclid. In America, many of the best teachers have long since deviated from the abstract Euclidean form, and while not failing to reach the goal of strict proof, the stiff Euclidean mould has been much modified; its dogmatic method has been to some extent supplanted by a heuristic treatment; the pupil has been led to think out at least a considerable number of proofs for himself. But there are, no doubt, still far too many classes in geometry, and possibly even some text-books, in which the proofs well deserve the epithet "mouse-traps." In England, recent actions of the examining bodies have opened the door for a far more free treatment of geometry in the schools, and a host of text-books, based upon observation, measurement experiment, have appeared.[8] In the best of these texts, strictly logical proofs receive due attention.

The Formal Side of Demonstration.

It may be questioned whether it is best to insist from the outset that all proofs shall be presented in the form which long experience has shown to be the most compact, clear and comprehensive. True, one of the ends aimed at in the teaching of geometry is accuracy and clearness of expression. But the goal is not reached at the beginning. Let the mastery of the substance of a proof suffice at first, without delaying long to polish the form. That will come later. There must first be a body of material to polish, and premature stress upon the verbal form may hinder comprehension of the thought to be expressed. Not all the elegance and verbal accuracy that are to be attained later need be inflexibly required at first. Indeed, the attempt to force all proofs into the same mould may be abandoned without weakening the effectiveness of the proof itself; it may indeed rather be enhanced, provided clearness of view and of conviction be not sacrificed.

When a piece of land is to be sold, it is described and the price and conditions of proposed sale are stated informally in whatever phrases seems clearest and most suitable. After the essentials of the transaction have been agreed upon the whole is summarized for permanent record in a form (deed) in which all the explicit allegations and covenants are made which experience has shown to be requisite to cover the questions

[8] For example:

Godfrey and Siddons, *Elementary Geometry, Practical and Theoretical*, Cambridge, 1904.

Eggar, *Practical Exercises in Geometry*, London, 1903.

Warren, *Experimental and Theoretical Courses of Geometry*, Oxford, 1903, reviewed in *Bull. Am. Math. Soc.* 1904, pp. 504–510.

Harrison, *Practical Plane and Solid Geometry*, London, 1903 which may possibly arise as to the facts, obligations or validity of the transfer. Just so, mathematical forms of statement have been devised which record the facts of a proof, and make explicit mention of all the considerations needed to fortify it against any attack; but this record of the proof is made after the proof has been achieved, and the attempt formally to cover all these points in the very beginning may as easily tend to obscure the main question as it would to carry on all the negotiations for the sale of a piece of land in the precise phraseology of a deed.

This is one of the phases of work in which the teacher can allow himself much more latitude in the class-room than is permissible for a good text. The latter must, of course, present its proofs in a manner which is not only unexceptionable as to matter, but also which may serve as a model as to form.

Taking Propositions for granted.

The discussion of axioms has already brought out the wisdom and necessity of taking propositions for granted whose truth is sufficiently obvious to the pupil, even though they might be deduced from simpler assumption. At this point an opposite class of proofs may be mentioned which it is also a good plan to omit temporarily in the first view of the subject, namely, those which the teacher's experience has shown to be specially difficult for the pupil to grasp. With the clear statement that the proof is deferred, it is pedagogically sound to assume the truth of such propositions and use them freely. The strength gained from such use and from other work on the subject will enable the pupil later to master the omitted proof with ease.

It is also an excellent exercise for the pupil to make the genealogical tree of some particular proof; that is, to state the propositions on which it depends directly, then the propositions on which these depend, and so on, until the assumptions used without proof are reached. The numbers of the various propositions can be arranged in the form of a diagram exhibiting their interdependence to the eye.

Few teachers now teach geometry without asking their pupils to demonstrate a goodly number of propositions themselves. The degree of success will vary, but it is generally conceded that the attempt is always worth while. How best to guide pupils in their search for demonstrations thus becomes a question of the first importance to the teacher. It is doubly important when the whole instruction proceeds upon the plan that, as a rule, the proofs are not to be *learned* but to be *found*, either by the class and teacher in common discussion, or by the pupil working alone.

Methods of Attack.

No specific rules of procedure can be laid down, but a few suggestions may be helpful. First of all, a clear idea must be obtained of what is known and what is to be found in the problem in hand. An excellent plan is occasionally to take up these two questions only with respect to a list of propositions. During the course of the search for a proof, formal rehearsing of what is to be proved may often suggest something that may be tried next.

It is also a good drill to enumerate the properties of the figure which are known in consequence of the data: what lines, angles, triangles, are equal; what lines are parallel, perpendicular; what figures are of equal area; etc., etc. A list of the chief things to look for might be helpful in the pupil's hands.

There may also be made a (growing) summary of the results previously established, and this summary, this collection of tools, may be looked over to find which tool seems likely to do the work needed.

Several summaries would be still better, one in the order of development, another classifying the results according to subject matter, and a third, according to what they will do. Thus, in the third summary there would be recorded under one head the means for proving line segments equal, under another those for proving angles equal, under others those for proving triangles equal, for perpendicularity, for parallelism, etc. With his tools thus arranged in an orderly manner the pupil resembles a carpenter. If he wants to drive a nail, he selects a suitable hammer; if he wants to cut a board, he looks among his saws.

Summaries.

Many writers have enumerated the more important methods of proof with special reference to their use in the finding of other proofs. By far the most important of these enumerations is the work of Petersen,[4] originally written in Danish but translated into German and into French. This work will well repay careful study.[5] Besides the problems used to illustrate the theory, it contains lists of problems to be solved under each head, over four hundred problems in all, ranging in difficulty from very easy problems to problems like that of Malfatti: "To inscribe in a triangle three circles each tangent to two sides of the triangle and the other two circles." Of a similar character is the work of Alexandroff,[6] which has run through a number of editions in Russian, and has been translated into both German and French.

Books on Methods of Attack.

The book of Sauvage[7] is also interesting, though theoretic and covering a wider range. For those not able to read German or French, mention may be made of a little book by Loomis,[8] whose bibliography gives other English references on this general topic.

Definitions in Geometry.

Geometry is built up on the foundation of definitions and axioms. These have already been treated in other connections. The tacit use of a large body of assumptions (axioms) which the pupil accepts as true without proof, the study of the things themselves and then

[4] *Methods and Theories for the Solution of Geometric Problems of Construction*. The German translation published in Copenhagen, 1879.

[5] An English translation was also published (London, Low & Co., 1879), but it seems now to be out of print, and I have never seen it included in a second-hand catalogue.

[6] Alexandroff, *Problèmes de Géométrie élémentaire*, Paris, 1899; also in German, *Aufgaben aus der niederen Geometrie*, Leipzig, 1903.

[7] Sauvage, *Les Lieux géométriques en Géométrie élémentaire*, pp. 119, Paris, 1893.

[8] Loomis, *Original Investigation, or How to Attack an Exercise in Geometry*, pp. 63, Boston, 1901.

their description (definition), the attainment of formal precision of definition as a growth rather than the acceptance of definitions as artificial ready-made products at the beginning, all these things have already been discussed. But that the definitions should not be administered to the pupil in ready-made form does not mean that the teacher should not have a clear idea of the geometric concept which he wishes the pupil first to have, then to describe. Quite the contrary.

In this connection a class of definitions may be mentioned which, as commonly given, would seem to admit of improvement. The class is sufficiently exemplified by the term *circle*. As frequently defined in a geometry class (but nowhere else) a circle is a plane disk, bounded by a curve, every point of which is equidistant from a fixed point. Everywhere except in the class in geometry the circle is the curve. There is no reason apparent, except custom, why the curve should not be called circle from the outset. The curve is less complex; the image of the curve must be produced before that of the surface enclosed by it has any meaning. The term *circle* is seldom used consistently in the disk sense, though some careful teachers and writers do speak of "passing a circumference through three given points," "an arc of a circumference," etc. This is logical when the circle is defined as a surface, and no better evidence of the change of meaning which the pupil must give to the term could be desired. It is out of harmony with common parlance and later mathematical usage. It may even very easily happen that the same teacher may teach the same pupils on the same day two different definitions of the same word.[9] The confusion is quite unnecessary and should be avoided. The same general remarks apply to figures of

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[9] "In mathematics a circle is a plane bounded by a curved line, all parts of which are equally distant from its centre. *In geography a circle is not a plane*, but is the line bounding it; the circumference of a mathematical circle." — Rand-McNally, *Gram. Sch. Geog.*, p. 11.

both plane and solid geometry. They are best defined as lines and surfaces respectively.

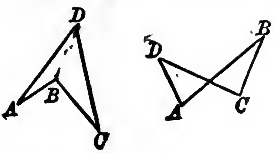
The definition of the circle as a surface no doubt had the area of the circle in view. But there is no trouble whatever in interpreting the "area of the circle" to mean the "area enclosed by the circle"; "volume of a sphere" as "volume contained by the sphere."

The notion of angle is somewhat analogous to that of straight line, so simple that it is difficult to describe it in simpler terms. The child has a working knowledge of what an angle is, and this may be used as basis for later extension of the idea. What he meets at the outset is the need to measure and compare angles. This can be done effectively by connecting the idea of rotation with that of angle from the beginning, a procedure which offers no difficulties in itself, but prepares the way for subsequent work in trigonometry and elsewhere.

Modern Geometry.

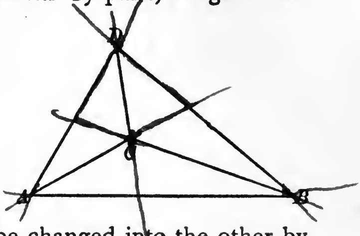
The researches of some of the leading mathematicians of the nineteenth century have vastly extended the subject matter of geometry, have introduced new and fertile methods and enriched the subject with many beautiful results. Some of these results and some of these methods are, so far as intrinsic difficulty is concerned, within the reach of secondary-school pupils, and the question arises as to the extent to which they may well be introduced into the instruction.

Modern research has established, for example, relations between theorems formerly treated independently, and has established methods and notations which permit the proof of more general theorems, which may include as special cases several of those heretofore taken up separately.



Thus, to be more specific, by introducing negative angles, and extending the definition of interior angles, a general theorem may be set up concerning the sum of the interior angles of a quadrilateral which shall apply, not only to ordinary quadrilaterals, but also to figures like those on page 270.

Likewise, by defining a quadrilateral as the system of all lines passing through four given points by pairs, a figure like the following would be a quadrilateral and would have among its properties those of the ordinary quadrilateral.



As a third instance may be mentioned the analogies which have been noted between pairs of theorems, such that one may be changed into the other by a systematic interchange of words, as:

for *point* read *line*,
" *line* " *point*,

and the like. The character of this relationship will appear from a few examples:

In a *quadrangle inscribed* in a circle, the sum of two opposite *angles* equals the sum of the other two.	In a *quadrilateral circumscribed* about a circle, the sum of two opposite *sides* equals the sum of the other two.
In every hexagon *inscribed* in a circle, the three *points* of intersections of pairs of opposite *sides* lie on the same straight *line*.	In every hexagon *circumscribed* about a circle, the three diagonal *lines* connecting opposite vertices (*points*) pass through the same *point*.
An *equilateral inscribed* polygon is regular.	An *equiangular circumscribed* polygon is regular.
An *equiangular inscribed* polygon of an odd number of sides is regular.	An *equilateral circumscribed* polygon of an odd number of sides is regular.

These instances serve as illustrations of more modern methods of treatment which do not seem beyond the grasp of secondary-school pupils in geometry, and writers of high repute both in Europe and America have introduced such ideas freely into works for beginners.

There is a strong reason to expect that a moderate and judicious use of the simpler methods of modern geometry would render the work in geometry clearer, more systematic, and more attractive; and few would claim that we should not utilize the methods of modern geometry when they would prove serviceable. The application of symmetry may be taken as an example.

Modern Geometry in the Class-room.

On the other hand, it is very questionable whether place can profitably be found for specific topics of more recent geometry, as, for example, the anharmonic ratio of four points on a straight line, or the modern geometry of the triangle. Here, as elsewhere, it may be asked whether the fact that the race did not easily or early discover these results does not betoken a certain lack of affinity between them and the human mind, a certain failure to stand out conspicuously, which has caused these results so long to escape the scrutiny of myriads of searching minds, and would continue to tend to prevent their easy ingress into the mind of the learner to-day.

The conservative course might seem to be not to introduce new *topics* from the modern work to any considerable extent, but to use its *notions* and *methods*, so far as they can be applied effectively, in simplifying or classifying the topics of the traditional list. The colleges of the country, while no doubt recognizing the value and elegance of these results in themselves, have not as yet included them in their requirements for admission.[10]

Teachers, on the other hand, will find few subjects of study more illuminating or helpful than that of modern geometry. Henrici and Treutlein's[11] work is specially to be commended as a strong treatise on elementary geometry with very free use of modern ideas. Casey's *Sequel to Euclid*[12] may also be men-

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[10] See in particular the Harvard requirements. See also *Report Com. Coll. Ent. Req.*, p. 143.

[11] Henrici und Treutlein, *Lehrbuch der Elementargeometrie*, 2te Auflage, Leipzig, 2 vol., pp. 146, 248, 1891, 1897.

[12] Casey, *A Sequel to the First Six Books of Euclid*, 5th edition, London, 1888.

tioned as a collection of propositions supplementary to the first six books of Euclid, constituting a good introduction to modern geometry. The material in German and French is extensive; an important work has been made accessible in English through Holgate's translation of Reye's *Geometrie der Lage.*[18]

The Parallel Axiom and the non-Euclidean Geometries.

Another line of modern research that enters the field of the subject matter of elementary geometry is that relative to the non-Euclidean geometries. These are based upon the non-acceptance of the statement that:

Through any point of a plane one, and only one, parallel can be drawn to each straight line not passing through the point.

Euclid himself seemed to recognize a difference in the degree of conviction carried to the mind by this statement and his other fundamental assumptions, calling the statement a *postulate*, rather than an *axiom*. But it was reserved for the mathematicians of the nineteenth century to recognize that there is no logical necessity to make this postulate at all; that it can be abandoned, replaced by some other postulate, and a logical system of geometric proofs still built up.

The geometries so reached will differ according to the hypothesis which replaces Euclid's postulate. In contradistinction to the geometry which retains the postulate, they are called non-Euclidean geometries. We have thus various geometries which, as Poincaré has well said, are not more or less true, but more or less convenient. In the material world in which we live, the Euclidean postulate seems to be satisfied; consequently, the Euclidean geometry will continue to be the geometry of practical life and hence of the schools, undisturbed by the discoveries resulting from acute speculations of mathematicians.

But this event of high logical import, the discovery and development of the non-Euclidean geometries, of which perhaps not even a syllable need ever be breathed in the secondary class-room, is still not without important pedagogic bearings.[14]

It settles once for all the basis upon which the Euclidean postulate must stand; it is an assumption pure and simple arising from concrete experiences, and like other axioms to be quietly assumed as evident and used as occasion may arise. It will warn the teacher from any attempt to *prove* the postulate, either for himself or his pupils — since there is no question that all such attempts must be classed with attempts to "square the circle."

The Use of Motion and of Signed Magnitudes.

Euclid attempted to avoid definitions and proofs based on motion, and his followers even down to modern days have followed his example to the detriment of geometric work. In recent times, however, there have arisen men who boldly make free and fundamental

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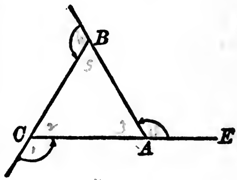
[14] "To be sure, as Study has well insisted, for a thorough comprehension of even the elementary parts of Euclidean geometry the non-Euclidean geometries are absolutely essential. But the teacher is teaching the subject for the benefit of the students, and it must be admitted that beginners in the study of demonstrative geometry cannot appreciate the very delicate considerations involved in the thoroughly abstract science. Indeed, one may conjecture that, had it not been for the brilliant success of Euclid in his effort to organize into a formally deductive system the geometric treasures of his times, the advent of the reign of science in the modern sense might not have been so long deferred. Shall we then hold that in the schools the teaching of demonstrative geometry should be reformed in such a way as to take account of all the wonderful discoveries which have been made — many even recently — in the domain of abstract geometry? And should similar reforms be made in the treatment of arithmetic and algebra? To make reforms of this kind: would it not be to repeat more gloriously the error of those followers of Euclid who fixed his Elements as a text-book for elementary instruction in geometry for over two thousand years? Every one agrees that professional mathematicians should certainly take account of these great developments in the technical foundations of mathematics, and that ample provision should be made for instruction in these matters; and on reflection, every one agrees further that this provision should be reserved for the later collegiate and university years." — Moore, *Presidential Address.*

use of motion, with happy results. See, for example, the text of Henrici u. Treutlein in German cited above, and in French, Méray, *Nouveaux Éléments de Géometrie.*[15]

Most pupils will find it decidedly easier, for example, to see that the three angles of a triangle make two right angles, by considering first the exterior angles, and note that the sum of the rotations indicated by each makes a complete turn, or four right angles. The pupil will have previously learned that the two angles at each vertex make two right angles, and that consequently the sum total of interior and exterior angles is six right angles. This proof is as rigorous as any that can be mastered by a beginner in geometry, and has the advantage of being readily made concrete. For example, a man stands at *A* and looks towards *E* (east); he turns to the left until he faces *B*; then walks straight to *B*; turns to the left until he faces *C*; walks to *C* and turns to the left until he faces *A*, then walks straight to *A*. In what direction does he now face? Through how many right angles has he turned?



This proof is easily understood, requires no explicit use of parallels, can come very early in the course, and at once opens the door for arithmetical and algebraic work. With the idea of motion comes that of *sense* of motion and of rotation. The generational notion of angle has no inherently greater difficulties than the ordinary one, and as soon as the barrier of prejudice against motion has been let down, the idea of rotation, its two senses, and of angles differing by complete revolutions all come marching in. All these signed magnitudes are constantly exemplified in material motions about us, especially in those artificially produced by man (*e. g.*, railroads, fly-wheels); on the other hand, they are the geometric equivalents of algebraic magnitudes, and they not only interpret algebraic results geomet-

[15] Dijon, 1903, pp. 450.

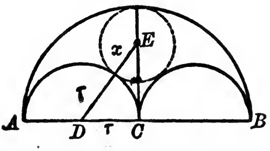
rically but also open up the possibility of attaining geometric results algebraically.

The fundamental notion of signed magnitudes is not the only close bond between geometry and algebra. Many a

Algebra in Geometry.

geometric problem permits its query to be formulated in algebraic language and the solution to be found by the processes of algebra.

For example: Given a line segment, $A\,B$ and C its middle point. On $A\,B$, $A\,C$ and $C\,B$ semicircles are erected, all on the same side of $A\,B$. To construct a circle touching the three semicircles



Let $x =$ radius of derived circle,
$r =$ radius of smaller given semicircle.

Then, from triangle $D\,E\,C$,

$$(r + x)^2 = r^2 + (2\,r - x)^2$$

Whence $x = \frac{2}{3}\,r$.

As a second example let us take the problem to draw a straight line which divides both the perimeter and the area of a given triangle into equal parts.

Let $A\,B\,C$ be the triangle; a, b, c the sides opposite the angles A, B, C, p the perimeter and $E\,F$ be the desired line. Let $A\,E = x$, and $A\,F = y$. Then, since the areas of triangles having a common angle are to each other as the products of the sides including the angle, it follows readily that $b\,c = 2\,x\,y$.

Further, from the conditions of the problem

$$x + y = \frac{p}{2}.$$

From these equations

$$x = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - \frac{b\,c}{2}}.$$

$$y = \frac{p}{2} \mp \sqrt{\frac{p^2}{4} - \frac{b\,c}{2}}.$$

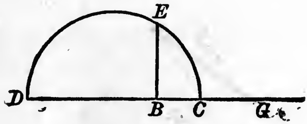
These values of x and y can be constructed geometrically as line segments, when b, c, p are given as line segments.

For, since the processes of elementary geometry permit of addition, subtraction, multiplication and division of line segments and the extraction of the square root, algebraic expressions made up of any combination of these operations can be constructed geometrically, by using line segments of lengths corresponding to the various numbers involved. This includes the geometric solutions of the quadratic equation.

Our particular quadratics above may be solved thus :

On a straight line, lay off $BC = a$, and $BD = \frac{c}{2}$ so that B lies between C and D.

On DC erect a semicircle. At B erect a perpendicular cutting semicircle in E.



Then $$\overline{BE}^2 = \frac{ac}{2}$$

With E as centre and radius $\frac{p}{2}$ draw an arc cutting BC at G.

Then $$BG = \sqrt{\overline{EG}^2 - \overline{EB}^2},$$
$$= \sqrt{\frac{p^2}{4} - \frac{ac}{2}}.$$

Having now a line segment to represent the radical, the roots themselves may be found by adding or subtracting this segment from the segment $\frac{p}{2}$.

This is a somewhat difficult illustration of the mode of constructing algebraic expressions. The pupil should of course begin with much simpler forms. For example, a, b, c, being line segments to construct:

$$ab,\quad \frac{a}{b},\quad \frac{ab}{c},\quad ab+c,\quad \frac{1}{c},\quad (a+b)c,\quad a^2b,\quad \frac{a^2}{b},$$

$$\sqrt{ab},\quad \sqrt{\frac{ab}{c}},\quad a\sqrt{\frac{b}{c}},\quad \frac{ab}{\sqrt{b^2-c^2}},\quad \sqrt{2a^2-3b^2},\ \text{etc.}$$

Problems of a more exclusively computational character, both algebraic and arithmetical, are also interesting and profitable, and very readily introduced.

For example, the area of a certain rectangle exceeds that of a square by 60 sq. in. The side of the square is $\frac{3}{4}$ of the shorter side of the rectangle, and the longer side of the rectangle exceeds the shorter by 16 in. Find the dimensions of the rectangle.

This leads to $$9x^2 - \overline{16}^2 x + 16 \cdot 60 = 0.$$

Each root should be tested to see whether it fulfils the geometric conditions of the problems, as well as satisfies the quadratic equation.

In addition to the point for which it was adduced, this equation also illustrates the simplifications of computation that very often arise from carrying along indicated numerical operations.

It is surprising that these types of interesting applications of algebraic results have not been more extensively used by American teachers and writers, in view of the considerable study of algebra (at least a year) that here usually precedes the beginning of geometry.

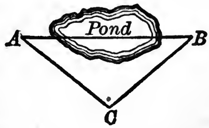
Notations.

It has been proposed always to use a single letter to represent a point, a line or an angle, using say capital letters for points, small letters for lines, and Greek letters for angles. Such notations would frequently have their advantages, yet might not always prove the best. To adhere to any one system of notation through thick and thin would prove hampering at times, no matter what the system might be. The class work especially should allow itself much more liberty in this respect than a text may. The use of different colors to mark auxiliary parts, to distinguish parts given from those to be found, and the like, is very advantageous. In one's thoughts it facilitates progress in many cases not to use any notation, but simply to think of the relation of "this line to that"; and it may be admissible sometimes, in the freedom of the class-room, to permit similar expressions, simply pointing and saying, "this angle is twice that," etc.

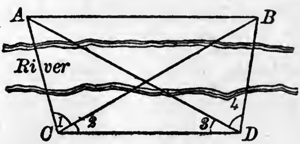
Experimental Work in Geometry.

Perhaps the primitive aspect of geometry as "earth measurement" has been too nearly lost from sight. Seldom does a class in geometry make any terrestrial measurements, and yet nothing else would give the subject so much life and reality. With a little improvised apparatus, two paper protractors tacked on a board, some wooden pegs and some cord, the pupils are equipped to make many interesting measurements. When thus the numerical side of geometry is not ignored, the door is opened to consideration of the numerical ratios of the sides of right triangles and thus quite naturally sufficient trigonometry may be developed for use with the field measurements. If the work in algebra runs parallel, enough of the theory of exponents might be given to permit the use of logarithms. But even without logarithms and without trigonometric ratios, a large number of interesting field measurements can be treated.

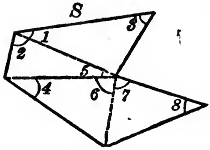
For example: To measure the distance between two points, which are themselves accessible, but are separated by some obstacle which prevents direct measurement of the line connecting them. Let the figure represent the conditions. Select a point *C* such that *A C* and *B C* can conveniently be measured. Also measure the



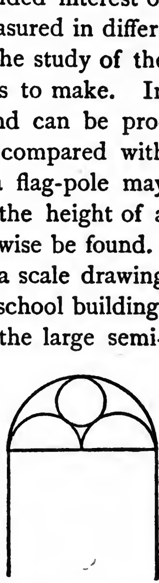
angle *A C B* — draw the quantities measured accurately to scale — the length of *A B* can be approximately determined from the drawing. Similarly, the length of an inaccessible horizontal object can be determined by measuring the line *C D*, and the angles 1, 2, 3, 4. Areas may



be likewise determined from field measurements, used as basis for accurate drawings. Thus the area of the irregular plot represented in the figure may be determined approximately if the side S and the angles marked 1 to 8 be measured. The whole would be drawn accurately to scale, the perpendiculars carefully erected, and the areas of the various triangles determined and added. The work brings different propositions and constructions into play and has the strong added interest of reality. The same area may of course be measured in different ways, and not the least interesting part is the study of the plot and the decision as to what measurements to make. In case a surveyor's map of the same tract of land can be procured, the maps made by the pupils may be compared with it. The height of the school building or of a flag-pole may also be determined from measurements, and the height of a visible but inaccessible vertical object may likewise be found.



Or the problem may be taken up to make a scale drawing of the window above the main entrance of the school building. If it is shaped like the figure, the diameter of the large semicircle can be found by measuring the width of the doorway at the bottom, and the other dimensions are deduced from this. The finding of the centre of the small circle is an interesting problem in itself, which we have already discussed. It arises naturally here as of practical consequence. These few instances may serve to exemplify the geometric problems in the real life at the pupil's door, whose solution will combine theory with practice in a way that is at once useful and interesting.



But field measurements are not the only available applications of geometry. Its range of applications is wide and its results can be made the basis of interesting experiments and measurements by the pupil without leaving the class-room. In the earlier years, work of this character has found more or less favor under the title of Inventional or Constructive Geometry, though none too much at best. But until quite recently the feeling seems to have been that demonstrative geometry had nothing at all in common with such work; that nothing of demonstration should be permitted in the former, and that all measurement and experimental verification should be forever discarded as an outgrown and useless garment, the moment the first demonstrations are undertaken.

Connection between Inventional and Demonstrative Geometry.

Of late, however, the more sound opinion has been gaining ground that these two ways of apprehending geometric truths may well be intermingled, that at bottom they have much in common, that the tyro in logical demonstration may often have his faith strengthened by some concrete verification or test of the results he has reasoned out, and that, on the other hand, measurements and experiments may often give him the clue to a demonstration he has been seeking or point out new theorems of which he had never thought before.

While recognizing to the full the need for careful demonstration, there is also in many quarters growing appreciation of the fact that unhesitating confidence in the operations and results of abstract logic is acquired slowly, and that for a long while constant recurrence to the experimental side will assure the beginner, and will contribute much to give him grasp of the essential facts and an ultimate mastery of abstract geometric reasoning, rather than to drive him to mechanical memorizing of uncomprehended jargon.

The close relationship between geometry and (mechanical) drawing is obvious. By arrangement between the teachers, the two subjects can be made to aid and supplement each other. The making of geometrical designs puts into practice the various constructions that are theoretically considered in geometry. Suggestive material will be found in books on geometrical drawing.[16]

Geometry and Drawing.

[16] For example, Hanstein's *Constructive Drawing; Geometrical Constructions.* Chicago, 2d edition, 1904.

In plane geometry the figures which are drawn are really models, since they represent very well that of which we are talking. In some instances other models are better which permit dissection or motion. All proof by dissection or superposition may be actually made by cutting up cardboard, while the pantagraph, for example, is useful in the theory of similarity, and various movable models[17] (linkages) both arouse interest and give opportunity to apply results obtained. For the transportation of figures, as, for example, in proofs of congruence, tracing paper may also be used.

Models and Apparatus.

In solid geometry the need for models is much greater than in plane geometry, where the figures are themselves actually models. Few unpractised minds can form an image of geometric figures without some visible aid. No one would teach plane geometry without constant use of drawings to represent the configurations under discussion, and many good teachers lay much stress on accurate drawing, in order to realize a close reproduction of the supposed conditions. In solid geometry, relations must be imagined which are more complex than those of plane geometry, and offer a distinctly new difficulty by extending into a third dimension. Many minds which are capable of grasping with ease the logical steps of a demonstration find difficulty in forming pictures of the configurations in space to which these demonstrations relate. Specially difficult to imagine are the plane sections of solids and the intersections of planes and lines. The training of the power of space intuition is quite another thing from the training of the power of logical demonstration.

Models in Solid Geometry.

Writers of texts have sought to aid by diagrams carefully drawn in perspective, by pictures with shading to give the

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[17] Details may be found in:

Kempe, *How to Draw a Straight Line*, London, 1877.

Dyck, *Katalog der Modelle*, u. s. w., Munich, 1895.

See also:

Klein, *Three Famous Problems of Antiquity*. (Transl. Beman & Smith, Boston, 1897.)

Row, *Geometric Paper Folding*, Chicago, 1901.

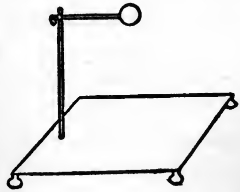
appearance of solidity and by photographs of models. All attempts, however, to represent three dimensional figures in two dimensions, must prove far inferior to an actual representation of the configuration itself in three dimensions.

Very simple and inexpensive materials will suffice for this purpose; some cardboard, paper, mucilage or passepartout binding, knitting needles and string will answer in most instances; in others, models may be cut from potatoes or turnips; an orange will serve to represent a sphere. Slits cut half-way across each of two rectangles permit them to be slipped into each other and to represent the intersection of two planes at various angles. Two circular disks may be used similarly to represent great circles on a sphere. It is best, of course, if each pupil has his own set of models which he has made himself, and which he can have constantly at hand. Not a little benefit is to be derived from the careful making of models.

Materials for Models.

Most classes will include some pupils who take special pleasure in making models, and who will gladly make the more difficult models, which may be preserved in the mathematical museum (p. 176). If funds are available, more expensive materials can be secured, out of which many models can be built. Hanstein's outfit [18] for this purpose is excellent. It consists of wooden rods and brass corners to hold them together, by means of which models of many of the propositions of solid geometry can be built up.

Apparatus for Models.

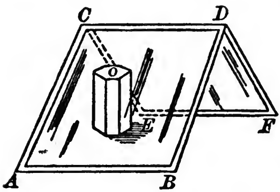


A frame made by Lepin and Masche, Berlin, has a bed of cork and a standard supporting a ball of cork. Lines are represented by rods, tipped with cork, and ending in needle points; the needle points can be stuck into the cork ball, the tips or the base, so as to form

[18] Made by the Randolph-Jones Mfg. Co., 243 Wabash Ave., Chicago.

any desired combination. A similar apparatus can easily be improvised, using soft wood instead of cork.

Another useful and very simple device consists of a hinged frame containing a pane of glass, *A B C D*. The frame is



set partly opened on a horizontal plane, and a geometrical model is placed within, as indicated in the cut. With a piece of chalk in a portcrayon, the pupil draws the outlines of the object as seen by him. A little practice of this sort gives him a much better idea than he could otherwise obtain of the way in which a drawing in a plane represents a three-dimensional object.

In schools having a manual-training department and workshop, a complete set of models can easily be prepared in the shop.

For the study of the sphere a slated globe is very desirable. Large slated globes, and also small ones, for individual use, can be secured through any good school-supply house. These houses also carry various models of the geometric solids, intended for very elementary work or for drawing. Of ready-made models for the propositions of solid geometry I know of none manufactured in this country except those of Baker,[19] who makes a very good set of about forty models for elementary solid geometry as well as a large number for more advanced work.

Solid Geometry and Plane Geometry.

It would be decidedly helpful if the study of the geometric solids and the training of the space imagination which is begun in arithmetic under the head of Form Study and Mensuration were not thereafter discontinued until the study of solid geometry is formally taken up. The space images which are already familiar need not be allowed to fade from mind; there are abundant opportunities to recall and refresh them in plane geometry, and opportuni-

[19] R. P. Baker, State University, Iowa City, Iowa.

ties can also be made in algebra. The faces of polyhedrons are plane figures which can be discussed, model in hand, in plane geometry. Still more important are the sections of geometric solids. Problems like the following may be admitted to the domain of plane geometry:

Find the surface of a cube of edge 5 in.; of a right prism of altitude 8 in. whose base is an equilateral triangle of edge 3 in.

Find the area of a section of a cube through two diagonally opposite edges.

Algebraic problems are also possible. For example:

The volume of a cube is 18 times the sum of the lengths of its edges. Find the length.

The breadth of a rectangular brick exceeds its height by 1 in., and the length exceeds the breadth by 1 in. The volume equals 48 times the breadth. Find the dimensions.

In short, without taking up solid geometry proper, the space intuition of the pupil may be frequently called into play in the work of the years that precede solid geometry. Whatever is based simply on the fund of ideas and facts which the pupil already has would seem legitimate in both algebra and plane geometry.

Some writers have attempted actually to interweave theoretic plane and solid geometry, taking up propositions as they related to the fundamental topic considered, without excluding either one, two or three dimensional figures.[20] While this plan has not as yet found wide-spread acceptance, the progress of the experiment will certainly be watched with interest.

The reasons for simultaneous treatment were well and conservatively presented by Veronese, at the International Congress of Mathematicians at Paris, in 1900.

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[20] *E.g.*, Paolis, *Elementi di Geometria*, Turin, 1884.
Lazzeri e Bassani, *Elementi di Geometria*, Livorno, 1898.
Veronese, *Elementi di Geometria*, 2d ed., Padua, 1900.
Méray, *Nouveaux Éléments de Géométrie*, 2d ed., Dijon, 1903.

"In instruction it is indeed fitting to go from the particular to the general, from the simple to the composite. Euclid, Legendre, and the majority of modern authors treat plane geometry completely before touching solid geometry. I think the beginning should be made with *Rectimetry* (geometry of the straight line), at least so far as relates to the principles. When the latter are well established, special theories, such as equivalence, similitude, measurement, etc., could be treated simultaneously in the plane and in space — I do not say, with the *fusionists*, that in these theories figures in three dimensions ought to intervene in the demonstration of planimetric theorems, but that numerous demonstrations can be extended simply from the plane to space. Such a mode of exposition will economize time and make the pupils grasp better the relations which exist between the diverse parts of the same theory. . . .

Reasons for Simultaneous Treatment.

"In this spirit my *Elementi* have been written. I announce first the postulates relative to the straight line considered in itself, and I define equality and inequality of segments without recourse to superposition. Thence I deduce the properties of addition and subtraction of segments. All these properties hold *mutatis mutandis* for a pencil of rays, for a circumference and for a pencil of planes. I give then the postulate which distinguishes a straight line from other figures, namely, that it is determined by two points. . . .

"By reason of the correspondence mentioned above between the straight line and the pencil of rays, the angle presents itself as a part of a pencil, just as the segment is a part of the straight line. The angle so defined must be distinguished from the *plane angle*, part of the plane bounded by two rays; the first is a linear entity, the latter a superficial entity. In the first edition (fuller, 1897) I demonstrate all the properties of a pencil of rays, corresponding to those of the straight line, considered in itself. In the second, I omit for brevity, certain of these postulates which are intuitive in the field of our observation. . . .

"The first three books concern general geometry of the straight line, of the plane and of space. In the fourth book I treat the equivalence of figures, plane and solid." [21]

[21] Veronese, *Report of Paris Cong.*, 1900, p. 446.

In this discussion of the teaching of geometry, the belief has been emphasized that geometry should be taught not as a collection of settled facts to be learned, even though the facts of geometry that are taken up in a first course have in the main been settled for thousands of years, but as a set of phenomena to be investigated scientifically. Geometry is a living and growing science; if it is taught so that the pupil himself makes some discoveries, he will feel this life; to make discoveries he must ask questions, he must scrutinize the various possibilities of the topic. Questions may be raised that are too difficult for an elementary course, or that open the door for some little account by the teacher of work in geometry, ancient and modern, beyond the scope of the course. The pupil will then come to the end of the course in geometry, with possibilities of study still unexhausted, perhaps with some problems still unsolved, and with hearsay knowledge of important lines of geometric study different from those he has followed. He will not regard geometry as a cast-iron subject whose sum total is recorded in the book he has studied, but as a large and growing field of which he knows a part; and he should look forward with pleasure to obtaining a deeper and more critical insight into the part he has already studied, as well as to extend his knowledge to other parts of the subjects. This anticipation would be appropriately met by suitable courses in elementary geometry in the earlier collegiate years.

The Close of the Course in Geometry.

The Teaching of Trigonometry

Trigonometry is usually regarded and taught as a separate subject, but when we come to seek its peculiar characteristics, we fail to find a clear-cut central idea which would serve to give the subject its own individuality. Arithmetic has to do with the number concept, algebra with the generalized number concept and the equation, and geometry with the space concept and its problems. These central thoughts are distinct and fundamental

The Teaching of Trigonometry.

and though the masses of material that their treatment has developed have many inter-relations and common border lines, still each of the subjects has its own very marked individuality.

What is Trigonometry?

What is the corresponding distinctive characteristic of trigonometry? Is it the study of the trigonometric ratios; *i. e.*, the ratios between the sides of a right triangle? That is a chapter of geometry. It has the earmarks of geometry, and nothing that would make it inadmissible in geometry. Is it the solution of triangles, as its name implies? That is a continuation and application of the study of the ratios, and also falls within the domain of geometry. Is it the manipulation of formulas and solution of equations involving the trigonometric ratios? This is algebraic in essence, and as soon as it is admitted that algebra may concern itself with numbers originally defined in geometry, the handling of trigonometric formulas and equations can find thoroughly appropriate quarters in algebra. Is there, finally, such coherence in the material ordinarily taken up under the title "trigonometry" that serious loss of unity would ensue if the various topics were taken up in connection with those portions of geometric or algebraic theory to which they are essentially related? By no means. What is ordinarily treated in elementary trigonometry falls of itself into a number of quite distinct parts whose relation to each other is by no means so close as to suffer if the parts are separated. As such parts there may be mentioned:

The Separable Parts of Trigonometry.

First. The definition of the ratios, their approximate values as found by measurement, the use of tables of the natural values, and application to problems that can be solved by means of the definitions only. All this work is merely a study of the right triangle, and finds its proper place in connection with the right triangle in plane geometry. Indeed, it may precede the Pythagorean proposition.

Second. The use and the theory of the use of logarithms. This is purely algebraic, and should be taken up in connection with exponents.

Third. The solution of general triangles, with and without logarithms; applications. All the formulas needed can be proved geometrically.

Fourth. The fundamental relations between the ratios; application of the theorem of Pythagoras; expression of each ratio in terms of the others; simple identities.

Fifth. Extended definition of angle; angle of any magnitude, positive or negative. Extension of the definition of the trigonometric ratios to these angles. Very simple equations, as $cos\ x = \frac{1}{2}$; find x (all the values). Identities.

Sixth. Formulas of the type $sin\ (270 - A) = - cos\ A$.

Seventh. Trigonometric equations.

Eighth. The addition theorem and its consequences.

These different parts have no very close connection. Their order can be varied, and they are more directly related to parts of algebra and geometry than to each other.

The first three or four are well within the reach of the secondary-school pupil, though the last four are perhaps more prudently left for collegiate work. Whether taught in connection with algebra and geometry, or separately, the matter of the first four topics is especially inviting for the secondary-school course.

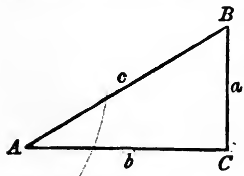
The Practical Interest of Trigonometry.

This material has the special merit of possessing simple and direct applications of a wide range and of obvious practical importance. Far more than any other subject within the possibilities of the secondary field, it enables the pupil to do things which he recognizes as of importance, and it gives him a sense of practical power through mathematics that may well be the climax of the secondary-school course in mathematics. To measure the distance between inaccessible objects, to determine the length and direction of an unbored tunnel, to see the most tedious and complex calculations he has hitherto met shorn of their difficulties and to see possibilities hitherto unknown opening up before him through the use of the marvellous device of logarithms, is a most striking revelation to the pupil of what mathematics will enable him to do. All this is also well within the reach of the secondary-school pupil's abilities; in fact, a rounded course in the elements of trigonometry can be given that is both easier and more interesting than the work that is usually done in solid geometry, and it is a pity that some simple work in trigonometry is not incorporated in the mathematics done by the mass of the pupils in the secondary schools.

A Simple Course in Trigonometry.

We pass to a more detailed consideration of such a course, if given separately. The fundamental proposition of trigonometry is the similarity of right triangles having one acute angle of each the same. This angle and the six ratios of the sides are seven quantities so related that when any one is given the others are fixed. The course may be begun by measuring these ratios and constructing the triangle when the angle or one of the ratios is given. For example:

Given the ratio $\frac{a}{c}$, to determine A and the other five ratios by measurement; or, given the angle A, to determine the six ratios by measurement. After some little practice with the ratios themselves, they might be given their customary names,



sine, cosine, tangent, etc., and in naming the co-functions their connection with the complementary angle may at once be noted. The consensus of modern opinion is that the trigonometric functions should be defined as ratios exclusively. There seems no special reason for making the hypotenuse unity, and in the particular cases where it may be desirable to do so no difficulty will arise under the ratio definition.

This may be followed by constructing angles varying by 5° from zero to 90° and determining by measurement the approximate values of their trigonometric ratios. These may be exhibited in a graph. At this point the tables of the natural values of the ratios may be introduced and used in the solution of problems. The use of the relations between the co-functions in abridging the table should be noted in passing.

With no more than the definition, interesting applications in the solutions of problems are at once possible, such as determining the projections of line segments, parallelogram of motions (various cases), composition of several forces, simple surveying problems, for example, the measurement of the height of an inaccessible vertical object; geometric problems, for example, the apothem, perimeter, and area of regular polygons. When feasible, data for problems may be secured by actual observation and measurement; for example, the mid-day angle of elevation of the sun may be determined by measurement of the length of the shadow of a perpendicular stick.

An algebraic phase of the work is the recognition of the fundamental relations between the ratios. These may be at once applied in the establishment of simple identities. At this point the theory and use of logarithms may be taken up and followed by a generalization of the definitions of the trigonometric ratios so as to include angles in the second quadrant. The formulas needed for the solution of all cases of triangles should next be developed. They can all be developed geometrically in a rather simple manner, and without use of formulas that are based upon the addition theorem. These would at once be applied to a wide range of applications of the general character of those that are already named. Thereupon the solution of trigonometric equations can be taken up.

The addition formulas and work based upon them are not necessary in secondary-school work, nor indeed for a complete treatment of the nominal chief end of trigonometry, namely, the solution of triangles, and are therefore with propriety deferred to a collegiate course, together with more elaborate formula work and so-called analytic trigonometry.

CHAPTER XIV

THE TEACHING OF ALGEBRA

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The Character, Scope and Relations of Elementary Algebra

The Definition of Algebra.

It is not an easy matter to define algebra, to draw a precise line of demarcation between it and some other domains of mathematics, in the elementary field between it and arithmetic, but fortunately it is also not an important matter.[1] The term falls under the head of those general terms (p. 191) which do not need exact definition in elementary work.

General Purpose and Character of the Teaching of Algebra.

The teaching of algebra is governed by the principles already discussed, especially in the consideration of the purpose and value of the study of mathematics, of the various methods, in particular the laboratory method, and of the relation between algebra and geometry, both in subject matter and in time of instruction. The general and most important questions having thus already been treated, there remain for consideration here only some special questions relative to the subject of algebra in particular.

Special Purpose of the Teaching of Algebra.

In addition to the general purposes of the teaching of mathematics, what special functions has the teaching of algebra? Four such functions demand special mention at the outset:

(1) To establish more carefully and to extend the theoretic processes of arithmetic.

(2) To strengthen the pupil's power in computation, by much practice as well as by the development of devices useful in computation.

(3) To develop the equation and to apply it in the solution of problems of a wide range of interest, including large classes of problems often treated in arithmetic, as well as problems relative to geometry, to physics and other natural sciences.

(4) To furnish such material within its domain as may be

[1] Several definitions that have been proposed are enumerated by Smith, pp. 162, 163.

needed in the later study of mathematics and of the various physical sciences.

These four functions are not divisions of the subject to be taken up and finished in turn, but goals towards each of which progress is continually to be made throughout the course in algebra.

Topics to be taken up in Algebra.

At the summer meeting of the American Mathematical Society in September, 1902, a special committee was appointed to prepare standard formulations of college entrance requirements in mathematics, in co-operation with committees already appointed by the Society for the Promotion of Engineering Education and the National Educational Association. The list of topics as formulated has since been adopted by various colleges as the statement of their entrance requirements in mathematics.

The following extract from the committee's report,[2] after defining the field to which the work of the committee was limited, gives the list of topics recommended in algebra.

"The committee understands its duties in the following sense:

"First: To specify those mathematical subjects which are generally recognized as appropriate requirements for admission to colleges and scientific schools.

"Second: To specify details under these subjects in such a manner as to represent the standards of the best secondary instruction — the word 'best' being interpreted in a qualitative rather than a quantitative sense.

"Third: The committee understands also that the consideration of pedagogic questions is not primarily among its duties. The committee is of opinion that no formulation should be considered as having more than temporary validity. No advantages attendant upon uniformity of definition could counterbalance any tendency of the definitions to retard progress of secondary education in mathematics. It is therefore recommended that if the definitions are approved, they be revised at intervals, perhaps of ten years.

[2] *Bull. Am. Math. Soc.*, November, 1903, pp. 74–77.

"*Elementary Algebra.* The four fundamental operations for rational algebraic expressions.

"Factoring, determination of highest common factor and lowest common multiple by factoring.

"Fractions; including complex fractions, ratio and proportion.

"Linear equations, both numerical and literal, containing one or more unknown quantities.

"Problems depending on linear equations.

"Radicals, including the extraction of the square root of polynomials and of numbers.

"Exponents, including the fractional and negative.

"Quadratic equations, both numerical and literal.

"Simple cases of equations, with one or more unknown quantities, that can be solved by the methods of linear or quadratic equations.

"Problems depending upon quadratic equations.

"The binomial theorem for positive integral exponents.

"The formulas for the nth term and the sum of the terms of arithmetic and geometric progressions, with applications.

"It is assumed that pupils will be required throughout the course to solve numerous problems which involve putting questions into equations. Some of these problems should be chosen from mensuration, from physics, and from commercial life. The use of graphical methods and illustrations, particularly in connection with the solution of equations, is also expected."

Classified according to subject matter, the topics may be arranged as follows:

Four fundamental operations.

Factoring (H. C. F.; L. C. M.; by factoring).

Fractions (including complex fractions, ratio and proportions).

Radicals (including square root of numbers and of polynomials).

Exponents (including the fractional and the negative).

Equations.

Of first degree, numerical and literal, one and more unknowns.

Of second degree, numerical and literal.

Of higher degree, if simple and reducible to preceding.

Applications of equations throughout course.

Problems from mensuration, from physics, from commercial life.

Use of graphic methods.

Other topics.

Binomial theorem for integral exponents.

Arithmetical and geometric progressions (*n*th term, sum) and applications.

Tendencies of the Day.

It will be noticed that the committee disclaims for its report anything of the character of an expression of opinion as to what *ought* to be the subject matter of algebra. The report is presented merely as a formulation representing the standards of the best secondary instruction of the day, and is based upon a large number of reports from men prominent in actual secondary instruction. The older current text-books may be taken to represent the standards of best usage at an earlier period, and which are still more or less influential. A comparison of these with the report of the committee will give some inkling of the direction in which the best usage of the day is tending.

One is at once struck by the fact that no new subject matter is included. Every topic named has long been included in elementary algebra. What is new is the stress on applications.

On the other hand, the list makes some noteworthy omissions. It does not call for H. C. F. otherwise than by factoring; that is, Euclid's algorithm is omitted. It does not call for treatment of ratio and proportion otherwise than under fractions. It does not call for cube root. (The inclusion of square root here, opens the door for its omission in arithmetic.) It does not call explicitly for simultaneous quadratic equations, and restricts the possibility of such systems to simple cases; it does not call for equations whose solution cannot be effected by means of equations of the first and the second degree, for general properties of equations, for indeterminate equations, for inequalities or for elaborate treatment of imaginaries. It restricts the treatment of the binomial theorem and the two progressions to the very simplest cases. In short, so far as such considerations as these may be trusted, the tendency of the best usage of our times seems to be decidedly in the direction of omitting the more abstract portions of elementary algebra and of giving more attention to the applications of the subject.

The Relation of Algebra to Arithmetic.

In many of its phases, algebra is only general arithmetic, and it is a great mistake to suppose that arithmetic is finished when algebra is begun. Indeed much of the theory of arithmetic can be taken up to advantage only after the use of letters is well in hand. It is possible to make the proofs without use of literal notation, but this simply means the considerable and quite unnecessary burden of carrying through the work a long locution instead of a single letter. The abridgment of the work in arithmetic has been urged on many sides and is being gradually put into effect. The *continuation* of the study of arithmetic as a part of the work in algebra is none the less important.

The Numerical Side of Algebra.

What has been said relates not only to the theory of arithmetic, but to its practice as well. Letters are not numbers, they simply represent numbers conveniently. All the operations of elementary algebra should be thought of as performed on numbers, not on the literal symbols for the numbers. After a time and gradually, this thought may fall into the background, when the character of the symbolism and of the operations with the symbols has been thoroughly grasped. But there should never be any difficulty to pass from the symbol to the thing signified (and for quite a while at least the meaning of the symbol should be kept constantly in the foreground). This may be achieved by continually replacing the letters which represent numbers by actual numbers. Teachers of physics often find that pupils see no numerical meaning in the formulas of algebra, that they are not able readily and properly to interpret and apply the formulas in numerical instances.

The numerical evaluation of algebraic expressions by the substitution of specific numbers for the letters cannot readily be overdone in beginning algebra. Later on, the solution of equations can be combined with the substitution. For example, fill the blanks in the following table:

Numerical Evaluation.

$$l = vt + \frac{gt^2}{2}.$$

l	v	g	t
——	5	32	6
100	2	32	——
150	——	32	2
70.4	3	——	2
——	−3	32.1	4

The explicit use of numerical instances will also do more than anything else to root out mistakes like the following:

1. $n(n+2) + 3(n+4) = (n+3)(n+4).$

2. $$\frac{2x^2 + 3x - 4}{2x^2 + 19x + 2} = \frac{3x - 4}{19x + 2}.$$

3. $$\frac{8a^2 \cdot 4c^2}{2} = 4a^2 \cdot 2c^2 = 8a^2c^2.$$

A formula is an algebraic sentence. The sentence should always have a meaning. The majority of the remarkable mistakes in transformation of algebraic expressions made by pupils, even after leaving the secondary-school class in algebra, are due to the fact that the expression and the transformation are meaningless jargon to the pupil.[8]

The use of specific numerical values for the letters is useful not only throughout the secondary school but far beyond it.

[8] Pupils say $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$ who would never say $\frac{1}{3} + \frac{1}{5} = \frac{1}{8}$. — Lodge, Teaching of Algebra, *Educational Times*, pp. 175–177, London, 1903.

Besides its utility in clearing up the pupil's conception of the process itself, it is also valuable as a check against mechanical errors and inadvertencies. If by an oversight the pupil writes $(2a + 3b)^2 = 4a^2 + 6ab + 9b^2$ he will detect the presence of an error by testing the correctness of the equation for $a = 1, b = 1$.

The practice also has the advantage of incidentally reviewing the operations of arithmetic. This latter is of prime importance. The numerical cases taken up in algebra should be sufficiently many, varied and difficult, not only to keep in practice, but to increase the facility in computation which the child brings to the high school.

Arithmetical Interpretation of Algebraic Results.

The algebraic formula is the source of many an arithmetical "short cut," as has already been pointed out. But the power to see this use can be attained only by much practice in replacing the algebraic symbols by specific values. For the pupil who has not been so trained,

$$a(b - c) = ab - ac$$

will contain no hint for the expeditious finding of 73×49, nor will

$$a^2 - b^2 = (a - b)(a + b)$$

help find 46×54.

As another illustration, let us consider the following mechanical rule for multiplying two numbers in which the number of tens is the same, and the sum of the units makes ten; multiply the units figures for the tens and units of the result, and one of the tens figures into the other increased by one for the hundreds of the result.

Example:

$$\begin{array}{r} 83 \\ 87 \\ \hline 7221 \end{array}$$

Process:

$$3 \times 7 = 21$$
$$8 \times 9 = 72$$

The algebraic formula on which this is based is:

$$(10a + b)(10a + \overline{10 - b}) = 10^2 a(a + 1) + b(10 - b).$$

Similarly, other rules for rapid calculation may be set up by means of algebraic formulas, and thus one of the uses of the subject exhibited which is always of interest to the considerable class of people to whom devices for abridgment of computation appeal. Rapid computers, whether only ordinarily skilful or remarkably so, use special devices adapted to the computation in hand.[4] These devices must be seen very quickly, and cannot always be provided in advance. All that can be done is to cultivate the habit of seeing the numerical content of a literal formula.

If the meaning of symbols be not kept in mind all sorts of errors result; for example:[5]

To show $(-a)(-b) = ab$. (a and b positive.)
Assumed: $(-1)a = a(-1) = -a$, for positive a's.
Let $-b = m$.
Then

$$\begin{aligned}(-a)(-b) &= (-a)m \\ &= a(-1)(m) \\ &= a(-m) \\ &= a\{-(-b)\} \\ &= ab.\end{aligned}$$

Here the writer deliberately throws sand into his own eyes.

The Meaning of Symbols must be kept in Mind.

He chooses the notation $-b = m$, apparently for no other reason than to hide the fact that m is a negative number. In the third line from the last he indeed shows that he has really thus blinded himself, and treats m as a positive number, thus begging the whole question.

Great care must always be exercised not to extend results or hypotheses to cases not included in the proof or assumption. This can only be done by keeping the meaning of the symbols clearly in mind.

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[4] Scripture, Arithmetical Prodigies; *Am. Jour. Psychol.*, 1891.

[5] From a text, see *Nature*, Vol. 59, p. 25.

The symbolism of algebra, indeed of mathematics in general, is a variety of shorthand, and it is necessary both to translate ordinary English into this shorthand, and also to translate the shorthand into English. Only by constantly requiring the latter can the teacher be assured that the pupils are really holding to the true interpretation of the symbols. Formulas should constantly be expressed in ordinary words and applied to special cases. Simple formulas from physics stated without proof are very useful for this purpose. Some examples will be given later in this chapter.

Translation.

A drawback to the study of algebra is that it lends itself readily to mechanical work. Pupils are so prone to form this habit that the teacher must be constantly active in keeping the meaning of the symbols prominently before the minds of the pupils. The difficulty is accentuated by the fact that a considerable portion of the work in algebra must needs be drill aiming at the attainment of mechanical facility in manipulating algebraic expressions. But the pupil who studies algebra is old enough and mentally sufficiently developed to understand the reasons of algebra as well as its routine. While the drill work of algebra needs much attention, it has perhaps been allowed to encroach unduly upon the phases of the work which require and stimulate thought. The mechanical manipulations are all compounded out of a few simple types. These types are readily mastered in themselves; their simple combinations can then be handled with equal readiness. These will suffice for the first year's work at least. Time will thus be gained, and what is more important, the deadening effects of mechanical grind will be minimized, if the pupil is not expected to plow through long lists of hard problems the first time he is taught some operation (multiplication, division of polynomials, use of exponents, for example).[6] When he can handle the simple cases with

Mechanical Work.

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[6] "It is quite true that algebra is in itself a purely formal science governed by a bare half-dozen of fundamental laws: these laws can be best acquired by the practice of purely formal opera-

ease and security he will have no difficulty in handling longer problems, done just in the same way, should he ever meet them.

The Central Topic of Algebra, the Equation

The Backbone of Algebra.

The list of topics that has been given (p. 295) is arranged according to subject matter. The more mechanical topics are put first where they should come in a topical analysis, but that by no means implies that they should come there in the class-room development. The central topic of algebra is, beyond question, the equation and its applications. It is this that puts flesh and blood upon the dry bones of the skeleton of algebraic routine, and the latter should not be developed all in a lump, but as needed for the solution of equations. Such processes, cases of processes, or complex instances as are not required in the solution of equations and problems, but still are needed in subsequent work, may well be deferred until later in the course. The amount of such material will be found to be quite limited. The needs of later mathematics will be much more satisfactorily met if the pupil has thoroughly mastered a few fundamental operations and formulas, and can apply them readily and accurately in such simple instances, than if, without this mastery, he is mechanically taken through all the cases and combinations that may arise. There is hardly time to do both. For example, in rationalization of denominators, the elementary course will have done sufficient if it has brought the pupil to understand the nature of the operation

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tions. But it is the teacher's duty to see that these mechanical calculations are only means to an end; and he must shun the temptation that so often presents itself to encourage the merely imitative instincts of his pupils at the expense of their reasoning faculties. It is comparatively easy to teach algebra in a way which earns marks in an examination, but is almost, if not quite, worthless from an educational point of view." — Matthews, *l. c.*, p. 191.

as exemplified in fractions of denominator $a + b\sqrt{c}$, or $a\sqrt{b} + c\sqrt{d}$, and has enabled him to rationalize these forms readily. Thus equipped, the pupil will be ready to handle the more complicated cases as he may meet them.

Omission of Long Drill Problems.

Exercises which have no value other than as "drill for drill's sake" are of doubtful utility. The subject will suffer no serious loss from the omission of complicated processes that are not likely to occur again either in future mathematics or in the physical sciences. Little harm would be done if, for example, problems like the following were relegated to the historical museum:

Find the greatest common divisor of:

$$6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1,\ 4x^4 + 2x^3 - 18x^2 + 3x - 5$$

Simplify: $\sqrt{(\sqrt[3]{9a^2b^2})^5} \cdot \sqrt[5]{(\sqrt[3]{125a^6 \cdot 25b^5})^4}$

Simplify: $\sqrt[3]{64} + 5\sqrt[6]{\frac{27}{64}} - \sqrt[3]{128} + 3\sqrt{343}$

Simplify: $\sqrt{7 + \sqrt{7 + \sqrt{7 + \sqrt{3,087,049}}}}$

Simplify:
$$\frac{\dfrac{1}{a-b} - \dfrac{a}{a^2-b^2}}{\dfrac{a}{ab+b^2} - \dfrac{b}{a^2+ab}}$$

These are not exaggerated illustrations; their prototypes can readily be found. The *processes* of algebra are not ends in themselves; they are tools needed in the accomplishment of the work of algebra, which is to solve problems; the problems may arise in mathematics itself, in the natural sciences, in technological work or in every-day life. One who omits from his algebra all processes not needed in any such problems will never feel the loss.

The goal of school algebra is the equation. It is by means of the equation that those problems are solved which are of interest and value in themselves, whether they relate to the domain of algebra, of geometry, of the physical sciences, or of practical life. The notations and the operations with the symbols have the solution of equations largely in view. Very little, indeed, of algebraic technique is needed for the solution of the simplest equations, and even these have interesting applications; consequently the practice is to be commended of carrying along work in equations, including applications from the outset.

The Equation.

The distinction between identities and equations of condition is very important. The identity states a *fact*, the equation of condition states a *problem*. The identity, $a^2 - b^2 = (a - b)\ (a + b)$, conveys a piece of information independently of the numbers that may be represented by a and b.

Equations of Condition and of Identity.

The equation (of condition) $x^2 + 3x = 4$, states a problem. It asks, "What number or numbers, if any, are there such that the square of the number increased by three times the number is 4?"

Equations of condition can be written at random as fast as the pen can fly. It does not follow that the problems which they propose admit of solution. Pupils are too apt to think that the mere existence of the equation (the statement of the problem) somehow proves the existence of the solution; they believe that the simple writing out somehow makes the numbers really and intrinsically equal. But it is quite possible to write out equations that propose problems which admit of no solution.

For example, there is no trouble in writing as a system of simultaneous equations,

$$2x + y = 5$$
$$2x + y = 8$$

But it is manifestly impossible that any values of x and y should exist such that $2x + y$ should be equal to 5 and also to 8.

Similarly the system

$$\begin{aligned} 2x - 3y - z &= 2 \\ 5x + 2y + 7z &= 4 \\ x - 2y - z &= 3 \end{aligned}$$

admits of no solution, as may be verified by attempting to solve it.

When numbers are intrinsically equal, as for example the areas of two squares of the same side, the equality may be seen by some form of comparison. There is no possible comparison that will establish any equality between $3x + 5$ as it stands and 14. It is an entirely different matter — a problem, not a fact — to determine under what *conditions* as to the value of x, $3x + 5$ amounts to 14.

Existence of a Solution.

The existence of a solution is often known from the concrete conditions which lead up to the problem. For example, "The perimeter of a right triangle is 30, and the altitude is 2 more than twice the base; find the dimensions." In this problem it is obvious to the geometric intuition that the triangle exists, and hence that the equation to which the problem leads must admit a solution. In other cases it is not evident that the concrete conditions can be satisfied. For example, "A has a square lot, B has a 25 ft. lot of the same depth. B's lot exceeds A's in area by 90 sq. ft. Find the dimensions of the lots in feet."

In elementary work the existence of the solution is always established by finding it, and the teacher should not press the question of existence on the pupils; the thing to be kept clear before them is the character of the equation as formulating a problem. Pupils often say, "It would not be an equation if it were not satisfied by some value of x." This is merely a matter of definition, and it is the definition of an equation that we are discussing. It is perfectly logical to agree that a formula in which the sign of equality connects two expressions involving an unknown quantity shall not be called an equation until we know that there exists a value of the unknown quantity for which the two expressions are actually equal. In this case, if we write at random,

$$x^6 + 7x^5 - 4x^4 + 2x^3 - 9x + 11 = 0,$$

we should not know whether or not what we have written is an equation until we have proved whether or not there exists a number which, when substituted for x, reduces the polynomial on the left to zero. The mere writing of the symbols does not assure that the value of x exists; does not make the formula an equation in this sense, any more than writing the statement "The moon is made of green cheese" establishes equality between the moon and green cheese. Whatever terms we use, the essential question is whether or not a value or values of the unknown quantity exist (root, solution) for which the two expressions are equal. Essentially, then, this type of equality proposes a problem, and it is equally logical as well as decidedly more convenient to call it an *equation* from the outset, and not to wait until we have solved the problem before doing so. Admitting imaginary numbers as solutions, it is capable of proof that all equations of algebra have solutions, and it is desirable that the teacher's knowledge of the theory of equations should be sufficiently extended to include this proof.[7]

Equivalence of Equations. When equations are regarded as problems, the subject of equivalence of equations is also made clearer. When two equations propose the same problems,—when, though perhaps knowing the solutions of neither, one knows that every solution of one is a solution of the other also,—the two equations are equivalent. Thus I know that any value for x which makes $x^2 + 7x$ equal to 23, will also make twice $x^2 + 7x$ equal to twice 23. I know, too, that any value which makes

$$x^2 + 7x = 23,$$

also makes $$(x + 2)(x^2 + 7x) = (x + 2)\,23,$$

[7] See, for example, Burnside and Panton, *Theory of Equations*, Fourth Edition, London and New York, 1899, Vol. I., pp. 259–261.

but I do not know that any value of x which makes the last pair of expressions equal also makes the first pair equal. In fact, it is not true; the equations are not equivalent.

Likewise, in the problem concerning the right triangle used above, the equation to which it leads,

$$x^2 - 44x + 195 = 0,$$

is satisfied by the values 5 and 39, of which only the first will verify the given conditions. The equation just written is not equivalent to that expressing the given conditions, from which it was deduced by a chain of operations, one of which is not uniquely reversible.[8]

The subject of equivalence should be clear in the teacher's mind, but need not be formally discussed in class except as the work of pupils may call for it. If any pupil finds a root which will not verify and is troubled by it, he is ripe for some explanations on equivalence of equations. This is simply a case of the general principle on which much emphasis is being laid nowadays (see Chap. VI.), that theoretic considerations should not be taken up separately for abstract discussion; but that some concrete case should lead the pupil to *want* to settle the abstract question involved.

The Study of Variation of Polynomials.

When once an equation is regarded as a problem, when $3x + 5 = 14$ asks for what value of x the expression $3x + 5$ has the value 14, it is very natural to ask how $3x + 5$ varies when x varies; among the different values of $3x + 5$, that of 14 may perhaps be found, and it is simply one of the many. This variation can be represented most clearly to the eye as a curve, and thus the geometric intuition can be utilized in the study of the properties of alge-

[8] "When one clears an equation of fractions by multiplying all across by some function of the unknown, the resulting equation contains other roots than the original one — yes, but it is not wise to trouble beginners with too much of this. One may philosophize deeply over our very simplest notions, but 'Sartor Resartus' ought only to be read by grown up people." — Perry, *Nature*, Vol. 63, p. 368 (1901).

braic polynomials. This has long been done systematically in secondary instruction in France, and its pedagogic value has recently been emphasized in England and America. (See Chap. VI.)

The Function Concept.

One of the possible discriminations between algebra and arithmetic is that arithmetic studies *values*, while algebra studies *functions*. Though, of course, this is not an exhaustive delimitation, it embodies an important point of view. The idea of functionality, of dependence of one variable quantity upon another, is in itself simple, and the actual existence of such dependence in the material world is a commonplace of the pupil's experience. For example, if a train moves at a uniform rate of speed, the distance passed over is dependent upon the time during which the motion has been in progress. In mathematical parlance: The distance is a *function* of the time.

The pupil who is beginning algebra can understand such dependence without difficulty, he can represent it graphically on squared paper, he will recognize the graph as the record of all special instances, will use it to solve particular problems, and will readily appreciate the connection between the uniformity of the variation and the uniformity of the rise of the straight line which represents it. Indeed, if not troubled with formalities and mechanism, his geometric intuition will lead him to feel the correspondence between the sameness of the *rate* of motion and the sameness of *direction* of the straight line.

In this connection the use of a single letter to stand for a polynomial, P, for example, may be connected, as well as $P(x)$, $Q(y)$, etc., to denote that the polynomial involves x or y, and also $P(5)$, $P(a)$ to indicate the substitution of 5 or a for x in the polynomial represented by $P(x)$.

General Equations.

The idea of general equations should not be urged prematurely. After considering a number of special cases, the pupil will come to it naturally himself. It will then be a relief to find that he can see all the essential properties of a class of equations without deciding beforehand which one he is talking about. After he has solved a number of problems differing only in numerical data, he will see, or be led to see, that they all are in reality the same problem, and that he can solve them all at one stroke if he is a little less specific as to the numerical values.

The general equation is like a blank draft or mortgage. The literal coefficients represent blanks to be filled out at convenience, but in the equation, as in the mortgage, all its characteristic properties can be seen as well, or even better, before the blanks are filled out as afterwards.

Putting into Equations.

It is a common experience of teachers that pupils find great difficulty in translating into equations conditions stated in words. Yet ability to do this well is one of the most important and valuable results of the study of algebra, the thought power so developed is one of its most useful products, and the pupil should not be allowed to end the study of algebra without a goodly measure of success in such translation. There is no royal road to skill in this process, but here, as throughout mathematics, the battle is almost won by the mere separation of the difficulties, and the victory is completed by a careful gradation of the instances under each type.

Thus many problems may be given requiring merely that a relation given in words be stated in the form of an equation. These problems may begin with the very simplest and increase in complexity very gradually to the most complicated statements involved in any problem to be given. At first instances may be used as simple as the following, or even simpler if need be; the teacher can give such problems off-hand until the pupils have all acquired facility and quickness in answering the questions:

If Frank is twice as old as Henry, and if A represents Henry's age, what represents Frank's age?

If John rides his bicycle three times as fast as William walks, and if t denote the time in which John rides a mile, what represents the time in which William walks a mile?

State in an equation that a certain express train runs four times as fast as a certain freight train.

The preceding statements are hypothetical. Others may be given from the natural sciences, but the pupil need not be able to prove them; he need not even know the meaning of the term used. For example:

State in an equation that:

1. The number of feet through which a body falls is thirty-two times the square of the number of seconds during which it falls.

2. The number of heat units generated per second by an electric current is .24 of the product of the resistance of the conductor and the square of the strength of the current, expressed in suitable units.

This work must not all be done at one time, but wherever new verbal forms of statement arise, separate drill on translating these statements into equations should be given. The drill on statements would be directly preparatory to the solution of problems involving them, and even in the complete solution of problems, the making of the statement should be kept a distinct part of the solution. It is equally important, conversely, to state in words the content of equations and formulas, and this likewise needs continual and gradual drill. Many pupils permit themselves to make statements in the equational form without consciousness of their meaning, and which they recognize as arrant nonsense when translated into words for them by their teachers.

The following may serve as specimens of types of problems useful for drill:

If v represents the velocity per second in feet, and if l represents the number of feet moved over in t seconds, what does $l = v\,t$ say? What does $v = \frac{l}{t}$ say? $t = \frac{l}{v}$?

If a body moves at the rate of 6 feet per second, how far will it move in 7 seconds? How long will be needed to move 48 feet? If it move 56 feet in 8 seconds, what is the rate per second?

If v denote the velocity per second in feet at the beginning of the motion, if a denote the increase in velocity per second, and V the velocity after t seconds, what does $V = v + at$ say?

If a body moves so that the distance l covered in t seconds is given by the formula $l = 3t + 4t^2$, state the law of its motion in words. How far would it move in 3 seconds? How far in the third second?

If D represents the density of a body, m its mass, and v its volume, what does $D = \frac{m}{v}$ say?

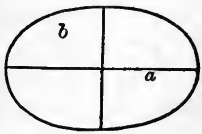
If t denote the number of seconds required by a pendulum to make a swing one way, if l denote its length in feet, what does $t = \pi\sqrt{\frac{l}{32}}$ say?

If g denotes some fixed but unknown number, what does $t = \pi\sqrt{\frac{l}{g}}$ say?

How could the value of g be determined experimentally by using a pendulum of known length and observing the time required for one swing?

If r denote the radius of a sphere, and v its volume, what does $v = \frac{4\pi}{3}r^3$ say?

Curves of a certain variety, looking somewhat like the illustration, are called *ellipses*. The two lines are called *axes*; a is the *semi-major axis* and b the *semi-minor axis*. If A denote the area enclosed by the curve, what does $A = \pi ab$ say?



If P denote the pressure to which a gas is subjected, and V its volume under that pressure, what does $PV = 1$ say?

This constant passage back and forth from the compact form of the equation (an algebraic shorthand) to the expanded verbal form will consume an appreciable fraction of the time, but much of it will be regained by the rapid solution of the problems themselves. In case of need, however, no hesitancy need be felt in cutting down the time allotted to the manipulative portions of the subject.

Solution of Problems.

In the solution of problems the following general plan may be followed:

I. Read the problem carefully and decide:

1. *What is given.*
2. *What is sought.*

II. After this reading, suppose the answer known, and represent one of the quantities to be found by some letter (usually x).

In the simpler problems it is at once evident what quantity should be so represented. In the more difficult ones there is room for choice, and the *ease* of the solution is sometimes dependent upon this choice; but the solution is always possible, no matter which one of the unknown is represented by a single letter.

III. Next, determine by what algebraic expressions the other unknown quantities must be represented, when the one selected is represented by x.

IV. Look for any *words* which contain in some form or other (expressed or implied) a *statement of equality.* These words must be replaced by the symbol = in the algebraic language, and the two things whose equality is stated by the words must also be expressed in algebraic symbols, and will then constitute the two members of an equation.

V. Solve the equation.

VI. Verify the result.

To summarize what has preceded, the equation is simply a statement, in the more compact language of algebra, of the fact stated in the problem in the more diffuse vernacular. The advantage of the algebraic form of statement is that, by means of its compactness, precision, and clearness, the existent relations are more readily seen, and from them equivalent relations are deduced, which lead to the determination of the values of the unknown quantity that satisfy the given conditions.

The Language of Algebra.

The following example from Sir Isaac Newton's *Arithmetica*

Universalis may be quoted in closing, the English translation being added here:

Mercator quidam nummos ejus triente quotannis adauget, demptis 100 lb. quas annuatim impendit in familiam, et post tres annos fit duplo ditior. Quaeruntur nummi.

(In English. A certain merchant annually adds one third to his money, diminished by £100, which he spends each year on his family, and after three years he is twice as rich as at first. Find the amount of his money at first.)

LATINE	ALGEBRAICE	IN ENGLISH
Mercator habet nummos quosdam	x	A merchant had a certain amount of money,
Ex quibus anno primo expendit 100 lb.	$x - 100$	Of which he spends £100 during the first year
Et reliquum adauget triente	$x - 100 + \frac{x - 100}{3}$ *sive* $\frac{4x - 400}{3}$	And increases the remainder by one third.
Anno secundo expendit 100 lb.	$\frac{4x - 400}{3} - 100$ *sive* $\frac{4x - 700}{3}$	The following year he expends £100
Et reliquum adauget triente.	$\frac{4x - 700}{3} + \frac{4x - 700}{9}$ *sive* $\frac{16x - 2800}{9}$	And increases the remainder by one third
Et sic anno tertio expendit 100 lb.	$\frac{16x - 2800}{9} - 100$ *sive* $\frac{16x - 3700}{9}$	And likewise in the third year he expends £100
Et reliquo trientem similiter lucratus est	$\frac{16x - 3700}{9} + \frac{16x - 3700}{27}$ *sive* $\frac{64x - 14800}{27}$	And similarly adds one third to the remainder
Fitque duplo ditior quam sub initio.	$\frac{64x - 14800}{27} = 2x$	And has become twice as rich as at first.

Latin	English
Quaestio itaque ad aequationem $\frac{64x - 14800}{27} = 2x$ redigitur; cujus reductione eruendus est x.	The question is thus reduced to the equation $\frac{64x - 14800}{27} = 2x$; from whose solution x is to be found.

A very important phase of the solution of problems, though the most difficult step of all, is the discussion and interpretation of the results. The distinction must not be overlooked between satisfying the mathematical relations implied in the problem and satisfying the concrete conditions there given. Both from its difficulty and from the degree of advancement of the material in which need for discussion arises, this phase of the work is deferred until very late in the course, but there it needs careful attention.

Discussion of Results.

One of the most efficient algebraic tools ever forged is the subject of determinants. Its field of application, systems of linear equations, is a part of the domain of elementary algebra, and the teacher who is acquainted with determinants, as a really well equipped teacher should be, may feel tempted to introduce some of his knowledge into his work with his class. But this would seem to be a mistake. It is hardly possible to make any formal use of the subject that will be clear and valuable, though informally the symmetry of the solutions of two linear equations with two unknowns may be pointed out, and a mechanical rule of cross products for indicating the roots by inspection may be set up.

Determinants.

Miscellaneous Points

The uses of oral algebra seem to have been largely overlooked or underrated. There is no reason why oral work should not play as important a part in algebra as in arithmetic. The development of much of the theory may be accomplished by oral work with very simple data, as well in the one subject as in the other. Facility of computation can also be attained by much drill on oral manipulations of simple expressions. On entering collegiate work high school graduates uniformly prove deficient in this respect; they have entirely too much recourse to paper, and write out very simple steps which ought to be performed half automatically in the mind alone.

Oral Algebra.

In arithmetic, number (whether denoted by the symbols of the arabic notation or by letters) means simply "how many," absolute quantity, and the numbers of arithmetic are therefore often called *absolute* numbers.

Extension of the Number System.

But it is easy to think of concrete conditions under which numbers, in addition to their absolute value, have a *sense*, one of two senses. Perhaps the simplest cases are those in which the numbers represent distances on a straight line or time intervals.

Consider the problem: "A, B, C, are three towns on a straight railroad. A traveller goes from A to B, 50 miles, then from B to C, 20 miles; how far is he from his starting-point?"

The problem is not definite unless we know whether or not in going from B to C he goes in the same *sense* as from A to B.

In this manner, and with other customary illustrations, debts and assets, income and expenditure, rise and fall of temperature, pull up and down, and the like, the idea of relative numbers — of numbers having one or two opposite senses — could readily be concretely developed in connection with the work in arithmetic, and the addition and subtraction of such numbers considered. But in accordance with the governing thought of much that has already been said in various connections, it seems better not to do so, because these numbers are not *needed* for the work in arithmetic, would lead to nothing in particular, and because the difficulty of *operating* with these generalized numbers is quite considerable to the pupil, and may well be separated from the first difficulty, — the use of the literal notation.

The fundamental operations with literal expressions and the equation should be used freely in arithmetic, but need be used only in concrete cases where the letters represent absolute numbers, and the numerical values are such that solutions in absolute numbers always exist.

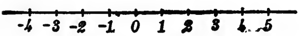
But in algebra the general question may not be overlooked of whether the idea of number may not be so generalized that the symbol $a - b$ shall always represent a number, whatever numbers a and b represent (the pupil can readily be led up to this by concrete examples). Much graphic work may well be the foundation of the development, motion, vectors, abscissas, and the idea can be made so concrete that no serious difficulty will be found in the notion of negative numbers or their addition and subtraction.

Nature of Positive and Negative Numbers.

Pupils should see clearly that numbers are not inherently positive or negative, but that the terms are relative, denoting two classes of number of opposite "sense," — that both classes must exist or neither, and that they may be interchanged, the choice of which shall be positive and which negative being a matter of convenience. If, from the wording of the problem or otherwise, one has an expectation of the sense of the result, there is a tendency to take that sense as positive. This is due in part to the name (a misnomer) and in part to the preference given to positive numbers by understanding numbers to be positive when no sign is indicated.

Numbers "Less than Nothing."

A number b is said in mathematics to be less than a number a if there exists a *positive* number c such that $b + c = a$. This definition includes the every-day concrete idea with which the pupils are familiar; it also defines when a negative number is "less" than another. If the numbers are arranged on a straight line,



every number is less than each number to the right of it. According to the definition, every negative number is less than zero. This says simply that if I go in the negative sense a certain distance, I must go in the positive or increasing sense to get to the starting-point. There is no mystery here, only the most banal truism; it is mentioned simply as a warning against the use of paradoxical expressions like "less than nothing."[9] When occasion arises, the technical definition "less than" should be deduced by the pupil from the relation of positive and negative numbers ranged along a straight line.[10]

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[9] This expression does not trouble pupils only. The "mystery" of the expression "less than nothing" is the sole *raison d'être* of a witty and well-written dialogue under the title, Algebra *vs.* English, *Education*, 1901–2, pp. 28–38.

[10] "The reasoning here employed pervades the whole of alge-

The names "positive" and "negative" are somewhat misleading; one class of numbers is not more "positive" than the other; distance north is not more positively a distance than distances south. The terms "additive" and "subtractive" would be more descriptive, though "jabberwock" and "jabberwee" would also answer. Positive and negative numbers are concrete, they are denominate or named numbers, just as much as 3 ft. or 7 in. We are considering two sorts of numbers: $+3$ means three of one sort; -3 means three of the other sort. The arithmetical numbers are the absolute or abstract numbers. Five is simply a number; it may be used as addend, subtrahend, multiplier, divisor, or otherwise; $+5$ is a number to be added; -5 is a number to be subtracted. The cases correspond to $5, $5 assets, $5 debts.

Positive and Negative Numbers.

The pupil should also see that even when the positive (and with it the negative) sense has been fixed, the same point may lie in the positive or negative direction from the origin from which distances are measured (the zero point), according to

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bra. A certain formula is shown to be true under certain restrictions as to the meaning of the symbols employed. If by an extension of the meaning of these symbols, the formula can be made to retain its validity, and if the extended interpretation does not involve any logical inconsistency, we are justified in making the extension, and in a manner compelled to make it by the demands of the calculus itself. That a formal science like algebra, the creation of our abstract thought, should thus in a sense dictate the laws of its own being is very remarkable. It had required the experience of centuries for us to realize the full force of this appeal, and it is therefore unreasonable for a teacher to expect his pupils to appreciate it all at once. A certain amount of compromise appears to be inevitable. Just as practical geometry may fitly precede the systematic study of the science, just as the experimental demonstration of physical laws helps to the comprehension of abstract dynamics, so the practical application of the laws of algebra before their logical necessity is fully realized is not only harmless, but even helpful towards the complete understanding of the very abstract considerations upon which their general validity is based." — Mathews, in Spencer's *Practice of Teaching*, pp. 184–185.

the position of the zero point. Thus, on both the Fahrenheit and the Centigrade thermometer, the positive sense is that of increase of heat, yet the temperature 23° F. is − 5° C.

Multiplication of Negative Numbers.

Multiplication of negative numbers offers more serious difficulty. The best procedure is no doubt simply to *define* the product of two relative numbers as the product of their absolute values, taken with the positive sign if both factors have like signs, and with the negative sign if they have unlike signs.[11] The definition may be justified by showing that it is consistent with previous definitions, and that under it expressions like

$$(a - b)\ (c - d) = ac - ad - bc + bd$$

always verify, whatever values are used for *a, b, c, d.* The so-called "proofs" amount in essence only to this.

Factoring.

The subject of factoring is important, but it is not treated best by lumping all that is said about it in one chapter. It has many connections, — for example, multiplication, division, fractions, radicals, solution of equations, — and should be treated in all these connections as needed and in distinct subordination to its applications. Time may be more profitably spent than on elaborate rules for factoring cases which will never arise elsewhere in the whole course.

It is easy to overestimate the importance of factoring and to give it more prominence in the class work than it deserves. Factoring, like the other technical operations of algebra, is not an end in itself, but an instrument which is valuable in the solution of problems, and the extent to which factoring is taken up may well be determined by the extent to which factoring is actually used afterward rather than by an attempt to treat completely all cases of factoring within a certain range of complexity. The pupil can go far in mathematics and its applications if he has thoroughly mastered a few simple types of factoring.

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[11] This is done by a number of recent French writers; for example, Tannery (J.), Borel, Bourlet.

The "factor theorem" and the "remainder theorem" are useful in themselves and in connection with the question of linear factors; they are very important in the further study of equations, and may be taken up with interest and profit in the latter part of the course in connection with the theory of the quadratic equation.

It is almost superfluous to say that results to problems in factoring should not generally be given by teacher or text. The pupil can readily verify the results by multiplication, and the verification is a good exercise in itself.

The boundless number of ways in which every expression, even prime numbers, can be factored, should not be overlooked.

Thus: $7 = \frac{1}{2} \cdot 14 = \frac{2}{3} \cdot 42 \cdot \frac{1}{4} = \frac{1}{5x} \cdot 35x$, etc.

In drill exercises some tacit restrictions are usually placed on the nature of the factors, as that they should be integral, irreducible, etc. But in practice such restrictions are frequently not made; one may wish, for instance, to take out some particular factor, as $\frac{4c^2 - 1}{2a}$, from a given polynomial.

Exponents.

The laws of operation with positive integral exponents can be deduced from the laws of multiplication and division, but the meanings to be given to negative fractional and zero exponents cannot be deduced from them. They are *definitions* which we are led to set up in the particular form which is customary, by the desire so to define these new types of exponents that the laws of operation which were effective for the positive integral exponents should be effective for the new exponents also.

It must be noted that if fractional exponents are defined so that one of the laws of integral exponents holds, say,

$$a^m \cdot a^r = a^{m+r},$$

which can be done in one, and only one way, this exhausts the possibilities of what can be done towards defining the new exponents so that all the laws for positive integral exponents shall be effective for them also. They can be so defined that *one* of the previous laws holds; it must be *proved* that for this definition the other laws hold also, — for example, that

$$(a\,b)^n = a^n\,b^n.$$

This proof is not difficult, but formal and abstract. It offers an opportunity to show the pupil a well-rounded instance of a formal and complete algebraic discussion, but the pupil would hardly be ripe for it before the later years of the secondary school at the earliest.

Logarithms.

The theory and practice of logarithms is naturally a part of the study of exponents, and does more than anything else to give the subject of exponents a real meaning. It is a pity that the two are ever divorced. Some drill in manipulation of exponents in accordance with the laws of exponents is necessary, but the essence of these laws will be much more deeply impressed by using them to abridge computation. Thus, consider the following table of powers of 2:

$2^1 = 2$	$2^6 = 64$	$2^{11} = 2048$	$2^{16} = 65536$
$2^2 = 4$	$2^7 = 128$	$2^{12} = 4096$	$2^{17} = 131072$
$2^3 = 8$	$2^8 = 256$	$2^{13} = 8192$	$2^{18} = 262144$
$2^4 = 16$	$2^9 = 512$	$2^{14} = 16384$	$2^{19} = 524288$
$2^5 = 32$	$2^{10} = 1024$	$2^{15} = 32768$	$2^{20} = 1048576$

By use of this table the value of expressions like the following can be determined by inspection:

$$512 \times 2048 \qquad \sqrt{131072}$$

$$\frac{256 \cdot 16384}{262144} \qquad \sqrt[3]{262144}$$

$$\sqrt[5]{1048576}$$

$$(64)^3$$

$$(512)^2 \qquad \sqrt[4]{\frac{(256)^5 \cdot (524288)^2}{(32)^6}}$$

Such problems impress the great computational power gained in working with exponents, and lead naturally to the discussion and use of tables of logarithms.

In this connection the logarithmic slide rule may be explained and used. Cheap varieties for school use are manufactured by mathematical instrument makers, but the pupil can himself readily graduate one that will show how the slide rule works.

Slide Rule.

The last half-century has witnessed marked progress in the view of irrational numbers taken by mathematicians. Dedekind, Cantor, Weierstrass have set up careful definitions, which have marked the beginning of a new epoch in this field. These theories are good things for the teacher to know, and fortunately some of the writings on the subject are accessible in English,[12] but it is more than doubtful whether they can be presented in a sufficiently elementary way to be available in secondary instruction.

Irrational Numbers.

The idea of irrational numbers is introduced most simply in geometry in connection with incommensurable segments. The proof that incommensurable segments exist may be made geometrically. This shows that if a straight line be divided into unit segments, there are points of the straight line whose distance from a fixed point is not a rational multiple of the unit segment. That is, to every rational number there corresponds a point of the straight line, but not conversely.

The algebraic proof that 2 is not the square of any rational fraction $\frac{a}{b}$ is a little more abstract, but also within the comprehension of the secondary-school pupil in the later years.

On the other hand, rational numbers can be determined, approximating as closely as we wish to $\sqrt{2}$; that is, rational numbers whose squares shall differ from 2 by as little as may be desired.

Indeed, two sequences of numbers,

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[12] In English: Fine, *Number System of Algebra*, Boston, 1890, especially pp. 26–35; Dedekind, *Essays on Numbers*, transl. Beman, Chicago, 1901, pp. 1–27.

In German: Bachmann, *Irrationalzahlen*, Leipzig, 1892, pp. 1–27; Burkhardt's *Encyclopädie der Math*, Vol. I.

	1,	1.4,	1.41,	1.414,	1.4142,	1.41421, etc.,
and	2,	1.5,	1.42,	1.415,	1.4143,	1.41422, etc.,

can be determined such that:

(1) the square of each number of the first sequence is less than 2;

(2) The square of each number of the second sequence is greater than 2; and

(3) The sequences can be extended (by the ordinary algorithm for the square root) until corresponding terms are found whose difference shall be as small as may be desired.

As far as given above, the differences are:

1, .1, .01, .001, .0001, .00001.

Treatment along this line, while not overtaxing the pupil's powers, will give him some idea of the notion of irrational numbers. More of the modern theory than this it would hardly be wise to give under ordinary circumstances, but the teacher should make himself acquainted with it. For fuller discussion, see the chapter on Limits.

Manipulation of Formulas.

While skill in the handling and transforming of algebraic expressions is to be attained, it must not be overlooked that here, as in arithmetic, the most complicated processes are nothing but a succession of simple steps involving but one principle at a time, and that hence drill would better be given on these steps, first singly, then in easy combinations. Very complicated formulas rarely arise in the pupil's further study of mathematics or its applications. Such as the pupil may meet can be handled without difficulty, if he have really mastered the simple steps and their simpler combinations. Taking the removal of parentheses, for example; if the pupil knows how to remove a single parenthesis, he can remove any number of parentheses. The pupil is very unlikely, in his later study of mathematics and its applications, to meet anything more complicated than a parenthesis within a parenthesis. Nests of half a dozen parentheses are almost unknown outside of books on elementary algebra. All the more mechanical processes of algebra should be confined to the simplest instances, only sufficiently complex to show the real nature of the process, provided they are not omitted altogether. Under this head would fall greatest common divisor of polynomials, square root of polynomials, cube root of polynomials, simultaneous quadratics. Nothing more dreary or less fruitful can be imagined than monotonous grinding out of long lists of complicated problems of this character.

In passing, a word may be said about the reading of parentheses. To read $3a(b+c)$ as "$3a$ times the quantity $b+c$" is a loose use of a term already defined; b is a quantity as well as $b+c$; it is better to read: "$3a$ times the binomial $b+c$."

For expressions longer than trinomials, it would do to indicate the beginning and the ending of the parenthesis by some such expressions as "bracket," "bracket ended."

Algebra applied in Physics.

Teachers of physics usually find their pupils unable to apply as well as they should the mathematics they have learned. Of the topics of algebra, the pupils are found to be especially weak in interpreting formulas, in passing from literal to numerical expressions, in interpreting equations, in applying variation (direct and inverse), in ratio and proportion. This gives the teachers of mathematics the cue to pay special attention to these topics. The last three of these topics really fall under fractions and should be treated in a simple, straightforward manner, without a mass of terms like extremes and means, antecedents and consequents, and rules involving these terms. No theory whatever is needed except the plain theory of fractions. These topics all lend themselves especially well to concrete treatment, and may be made among the most interesting of the whole subject, while at the same time preparing pupils to meet the justifiable expectation of teachers of physics, that pupils coming to them be readily able to answer such questions as:

If $$t = 2\pi\sqrt{\frac{l}{g}},$$

how does t vary with l? with g? etc.

It would be well if the teacher of algebra would ascertain from the teacher of physics in the same school quite specifically what mathematical relations are to be discussed and what mathematical problems solved in the course in physics, together with the notation in which they are to be expressed. These could all be taken up in the same notation in the algebra work. There is no reason why the equations $s = \frac{gt^2}{2}$, $pv = c$ (Boyle's law), should not be treated in algebra as well as $y = ax^2$ and $xy = c$. If the pupil learns in algebra that an equation may be solved for v, or for t, or even for a, he will not, as some pupils actually do, first replace t or v by x, and then solve for x.

The possibility of expressing *any* quantity that occurs in an equation in terms of the others is important, and should be clearly grasped by the pupil.

Thus, from $$3x + 2 = 4b$$

we have not only $$x = \frac{4b - 2}{3}$$

but also $$b = \frac{3x + 2}{4},$$

$$2 = 4b - 3x,$$

$$3 = \frac{4b - 2}{x}, \text{ etc.}$$

Relation of Algebra to Geometry and Physics.

The relation of algebra to geometry and to physics is that of a set of tools, of which the equation is by far the most valuable. These subjects in turn lend interest to algebra and help justify its study. It is consequently natural that data for problems leading to equations be sought in geometry, in physics, as well as wherever they can be found in the pupil's environment. This, of course, is possible only if these subjects have been taken up simultaneously with the algebra or antecedently (see pp. 183–188).

An Outline Order of Treatment.

In conclusion, an outline order of treatment may be presented, embodying in a specific way some of the suggestions of this chapter.

Literal Arithmetic.

1. *Letters to represent numbers* (*Abbreviations*).

2. *Solution of equations* and problems leading to them; *easy* enough that the mode of solution may be evident without any theory, — *e. g.* $4p = 12$; $3v + 2 = 17$.

(Hereafter "Equations" shall be used to include "Problems leading to them.")

3. *Notation* $a\,x$; $4\,b\,c$. *Very simple exercises* — as $3\,a\,x + 5\,a\,x = -$; $16\,mp \div 8 = -$.

Meaning of formulas: e. g. If l denote the length of a building lot and b its breadth, what does $l = 4\,b$ say? What does $l\,b = 6000$ say?

Translation into formulas: *e. g.* State as a formula that the height of a statue is twice that of the pedestal. Write an expression for the whole height.

4. *The idea of balance.* What may be done without destroying the balance?

Equations solvable by the theory given.

5. Positive Integral exponents (no formal laws). Four fundamental operations. Multiplication and division by monomials only. (Factoring in connection with multiplication and division.)

6. Meaning and translation of formulas continued; arithmetical interpretation; "short cuts."

Evaluation of expressions.

7. Literal fractions (simple denominators, — usually monomial).

Equations and problems, including easy cases with two unknowns.

Algebra.

8. The idea of relative numbers. The number scale.

9. Some simple problems, with data like those previously had, but leading to negative results. Concrete interpretations.

10. The four fundamental operations reviewed and extended, covering also negative numbers. More difficult problems. Evaluation for negative numbers also.

Factoring in connection with multiplication and division as before. Translation into and from formulas as before.

Equations and problems, including negative data and results.

11. Fractions, reviewed and extended, covering negative numbers also. Equations and problems.

12. Summary of equations of first degree, in one and several unknowns.

13. Definition of square root. Extraction by factoring.

14. Solution of quadratic equations.

15. Quadratic equations and problems, from very easy to medium.

16. Positive integral exponents; laws; problems.

17. Negative and fractional exponents; laws, problems in sufficient variety to cover all cases usually treated under radicals.

Equations and problems.

18. Radicals. Notation; a few problems only, the main treatment of the subject being given under exponents. Some problems in both notations.

19. Equations solvable like quadratic; systems of quadratics.

20. *Theory of Quadratics;* imaginaries; discriminant; relation between roots and coefficients.

21. Square root and applications.

22. Arithmetical and Geometrical Progressions and applications.

23. Binomial Theorem (Positive Integral Exponents).

Remarks.

1. Under favorable conditions from four to seven of the first points can be covered in the grades. In this case they should be reviewed in the high school.

2. The completion of point 15 might mark the end of first year's work in the high school.

CHAPTER XV

LIMITS

Geometric Limits.

In elementary mathematics the subject of limits is usually taken up in geometry and applied to geometric problems, yet the values sought are numerical values, and the limits usually obtained are limits of sequences of numbers. Thus, when we say that the area of a circle is the limit of the areas of a sequence of inscribed regular polygons when the number of sides is increased without bound, the numerical measures of the areas in question are meant. The idea of a geometric limit, however, also occurs in elementary geometry. For example, the tangent to a circle may be regarded as the limit of a secant, when the two points of cutting approach coincidence. But it is not imperative to take up such purely geometric limits in elementary geometry, and it is no doubt wise to avoid them, as is ordinarily done. In the instance cited, for example, the definition of the tangent as a straight line having one, and only one, point common with a circle, serves the purposes of elementary geometry, and does not bring the idea of limits into the foreground.

Incommensurable Segments.

It is not so easy, however, to avoid the idea of limits in some of the mensurational propositions, unless the proof is to be obviously incomplete. The theorem of Pythagoras enables us to construct a square whose area is twice that of a given square. The sides of these squares are incommensurable. Using them as sides of a rectangle, the area of the rectangle cannot be found by dividing it up into unit squares, however small. Here the method of limits is usually brought into play to show that the area is nevertheless the product of the numerical measures of the base and altitude.

Let us examine a little more closely what this means. By hypothesis the base and the altitude have no common linear unit of measure. Therefore the numerical measures of these sides (which simply count how often a linear unit can be applied) cannot be expressed in terms of a common unit. What meaning has it then to speak of the product of these numerical measures? What meaning has three times $\sqrt{2}$?

It can easily be proved that no rational number exists whose square is 2, and that 3 and $\sqrt{2}$ have no common measure. To determine what their product is requires a more careful study of $\sqrt{2}$; of irrational numbers in general.

The Real Difficulty, Numerical.

We started out with a problem whose difficulty lay in the incommensurability of its data. The customary treatment by the method of limits abandons the problem with Q. E. D. at a point where we are still confronted by essentially the same difficulty, the incommensurability of the data. In what sense, therefore, is the customary treatment a solution? In what respect does it even advance us towards a solution of the real difficulty? The same question may be raised as to each of the usual propositions of elementary geometry, involving a commensurable and an incommensurable case. The customary treatment at most pushes the difficulty back from geometric measures to numerical measures.

Two alternatives are open: we must either make a careful study of irrational numbers or abandon the attempt to include treatment of the incommensurable cases in the course of geometry in the secondary school.

Omit Proof of Incommensurable Cases.

A decade or two ago advocacy of the second alternative would have sounded like heresy, but if the investigations of the last few years have taught us anything it has been that it is quite out of the question to make the treatment of geometry in the secondary school complete in all its phases and cases. The most that we can hope to do is to teach matter which shall be consistent with the more advanced theory, to avoid positive error, to speak the truth even though abandoning the profession of speaking the whole truth.

It will be in accord with this view frankly to confine proofs to commensurable cases only. This does not mean that the relations should not be used for incommensurable cases also, but that such use should be based upon the statement that the formula can be interpreted and proved for this case also, though the proof is omitted on account of its difficulty.

A second category of proofs in elementary geometry customarily treated by the method of limits relates to curved lines and surfaces. The questions are those of length and volumes; that is, questions of numerical measurement of given magnitudes.

The Area of the Circle.

The determination of the area of the circle may be taken as sufficiently typical. This involves the idea of the length of the circle. But the length of a curve is not easily defined. Our intuition regards the length of a curve as the amount of its extension, but this is very vague until we begin to measure this amount. This is done in the case of the straight line by applying repeatedly a certain unit segment, for example, one inch. In some cases the given segment is an integral multiple of the unit segment, or of a segment determined by dividing the unit segment into an integral number of equal segments. We shall call such a segment a fractional segment. In other cases the given segment is not an integral multiple of any fractional segment whatever. But in such cases a fractional segment always exists, such that the given segment differs by less than any specified amount (segment) from an integral multiple of this fractional segment. In instances like this there are thus two cases, one of which is straightforward and simple, and covers the ground so thoroughly that very few pupils would of themselves think of the possibility of another case.

The Problem Difficult.

In problems like that of the area of the circle, on the other hand, the difficult case is the only one. The difficulty of such problems may be estimated from the fact that though they were attacked by Euclid, and have constituted a part of the subject matter of geometry ever since, it was left for the nineteenth century to evolve a rigorous treatment. Teachers and pupils find these propositions difficult because they *are* difficult, as well as because the treatment that is usually given them lacks the same degree of precision and clearness that is found in other parts of geometry.

What can be done in the Class-room.

The average secondary school pupil is not ready for the consideration of the nature of irrational numbers, the meaning of the length of a curve, or the area of a plane surface bounded by a curve, or of a curved surface, or the volume of a solid bounded by a curved surface. It seems far more prudent not to raise any question as to the meaning of these terms in the first course in geometry, but to leave the pupil's naïve intuition of them undisturbed. The formulas needed may be determined approximately by some concrete procedure (measurement, weighing, or the like), with no attempt at formal proof. The instruction should make clear and utilize the idea of approximation, that of closer approximation by more delicate measurements, and that of approximation close at will if the process of measurement is (in fact or in thought) sufficiently delicate, but make no attempt at a more formal establishment of the formula. This would be an instance of letting the pupil use a watch before he knows how it is made, a procedure certainly to be commended. The formulas may be used freely, with confidence in their correctness, based on their approximate experimental verification, and the considerable amount of time saved by not plodding through the dreary and heavy treatment that is customary may be very profitably spent on less critical phases of the subject.

The treatment of these problems has been retained in the geometry of the schools mainly for two reasons:

Why a Treatment of Limits has been retained hitherto.

(1) The ideal of school geometry as a complete logical structure, in which no statements, except axioms, are to be used unless they have previously been strictly proved; and

(2) The belief that the customary treatment by limits constitutes a strict proof of the propositions in question.

In view of the light that recent researches have thrown upon this ideal and this belief, there would seem no longer to be any motive for endeavoring to prove these propositions in the course in elementary geometry, or, indeed, until the machinery that is specially built for handling such problems — the calculus — is taken up. The calculus is, of course, out of the question for the secondary school, but may be taken up in an elementary and concrete way in the Freshman year in college.

No extended treatment of the subject of limits will be undertaken here. A few remarks on the subject will suffice to support the opinion expressed above that the difficulties of the subject are such as to warrant deferring it to a later period.

Limits in the Domain of Rational Numbers.

The simplest case of limits arises when only rational numbers are considered. In this case the customary definitions of limit are readily understood and easily illustrated. Thus:

The limit of a variable is a constant from which the variable may be made to become and remain different little at will.

This definition is highly elliptical. The following is more explicit:

Considering a variable y *dependent upon a variable* x, *the variable* y *is said to have (approach) the limit* L *(a constant) as* x *approaches* A, *provided* y *can be made to become and remain different little at will from* L *by choosing* x *sufficiently little different from* A.

This definition is more explicit than the previous one, but it can be made more precise by the use of additional symbols:

Strict Definition of Limit.

Considering a variable y *dependent on a variable* x, *the variable* y *is said to have the limit* L, *for the value* a *of* x, *provided that to every positive number* ϵ *there corresponds a positive number* ∂_ϵ *(dependent on* ϵ*) such that* $|y - L| \angle \epsilon$ *for every* $x \neq a$, *such that* $|x - a| \angle \partial_\epsilon$. (The bars $|\ .\ .\ .\ .\ |$ are used to denote the numerical, or absolute, value of the number enclosed, irrespective of its sign.)

The last is a strict definition of the term "limit," and it states formally the fundamental ideas of the other phrasings. They all express mathematically the idea that underlies the use of the term "limit" in common parlance. Thus a buyer may have a limit as to price; the boundary of a building lot is the limit of the owner's rights. If I stand directly south of a solid stone wall, with no intervening obstacle, the wall is the limit of my ability to walk directly northward, etc. In all of these instances there exists a fixed quantity, to which a variable quantity may approximate close at will. What the fixed quantity is, is seen from the *law* of the variation rather than from any actual variation. The same ideas are found in mathematical limits.

Limits in Common Parlance.

If we consider the variable $7x$, dependent on the variable x, and ask for the limit of $7x$ for the value $x = 5$, we assert that 35 is this limit, for $7x$ may be made to differ little at will from 35 by taking x sufficiently little different from 5, and if x be taken different still less from 5, $7x$ will differ still less from 35. (It would be sufficient if $7x$ did not differ *more* from 35 than before.) The use of ϵ and ∂_ϵ serves to specify numerically what is meant by "little at will," and "sufficiently little." Thus, if $\epsilon = 2$, ∂_ϵ will be $\frac{2}{7}$. That is to say, for every value of x which differs numerically from 5 by less than $\frac{2}{7}$, $7x$ differs numerically from 35 by less than 2. Similarly, if $\epsilon = \frac{1}{1,000,000}$, ∂_ϵ will be $\frac{1}{7,000,000}$, and $7x$ differs numerically from 35 by less than $\frac{1}{1,000,000}$, for every x which differs numerically from 5 by less than $\frac{1}{7,000,000}$. Evidently a ∂_ϵ exists for every ϵ, that is, 35 is the limit.

Illustrations.

In this case the limit is the value which the variable $7x$ assumes when $x = 5$. But this need not be so. For example, if $I\,[n]$ denotes the greatest integer in the numerical value of n, and if

Limits and Values.

$$y = 2x + I\,[I\,[x] + 1 - x]$$

then for positive integral values of x, y will have a limit and a value, and these will differ. Thus, as x approaches 3, whether through values larger than 3 or smaller than 3, y approaches 6, but when $x = 3, y = 7$.

For any specification of equality or inequality, existence or non-existence of the value and the limit, instances can be given in which the stipulated conditions are satisfied. The question of the existence of a limit and what its value is, is then quite independent of whether or not the expression has a value for the specified value of x.

Limits of Indeterminate Expression.

A case of special interest is that in which a definite limit exists, but where the expression resulting from substitution may have any value whatever. An example will make the type sufficiently clear:

Let
$$y = \frac{x^2 - 16}{x - 4}.$$

As x approaches 4, y approaches the limit 8, but if 4 is substituted for x, $y = \frac{0}{0}$, an expression which can be proved equal to any selected value.[1]

Limits of this type of expressions are fundamental in the calculus, and the illusory character of the result of substitution in such an expression gives a strong impulse to examine its limit. The theory of limits would never have been devised for expressions of the type first illustrated ($y = 7x$); still such limits are of the utmost importance in determining the limits of expressions of the later type.

In what has been said, the variables involved have been free to take any rational value. There are, however, cases of limits in which not all rational values are open to the variables. For example, only the values

$$\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \tfrac{31}{32} \ldots\ldots \frac{2^n - 1}{2^n}$$

may be available.

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[1] To prove it equal to 7, note that $\frac{12}{4} = 3$ is true because, and only because, $4 \cdot 3 = 12$. Likewise $\frac{0}{0} = 7$ is true since $0 \cdot 7 = 0$. Quite similarly $\frac{0}{0}$ can be proved equal to any number.

If the terms of this sequence of numbers are numbered from the left, and if $y = x$th term, then y approaches the limit 1, as x grows large without bound.[2]

This example also serves to illustrate the case in which a limit exists for y, as x grows large without bound. There is no difficulty in rewording the definition of a limit in an alternative form covering this case.

Irrational Limits. Though quite a few of the characteristic processes and results of the calculus could be developed on the basis of the definition of limit discussed above, restricting the quantities to rational values, this basis would be quite insufficient for geometry where the limits sought are irrational. Geometry demands, first of all, that the definition be made to apply to irrational limits, and it is desirable that illustrations of irrational limits be given. One or two illustrations of rational limits (as that of y assuming in turn the value $\frac{1}{2}, \frac{3}{4}, \frac{7}{8} \ldots$) are quite inadequate as a basis for the treatment of the irrational limits of geometry. And before we can consider irrational *limits*, we must consider irrational *numbers*.

Irrational Numbers. What is an irrational number? How shall the square root of two be defined? When we first meet this concept, we know only rational numbers, integers and fractions. We have previously noticed that some of these, as 4, 9, 16, $\frac{1}{9}$, $\frac{4}{25}$, etc., are products of two equal factors, and have called one of these factors the *square root* of the product — thus:

$$\sqrt{16} = 4; \sqrt{\frac{4}{25}} = \frac{2}{25}, \text{ etc.}$$

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[2] This is very commonly the first and often the only illustration used in setting up the definition of a limit, the sequence of numbers being usually written in the form:

$$\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{4}, \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}, \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} \ldots$$

In setting up so delicate a definition as that of a limit numerous illustrations should be used to make the idea as clear and concrete as possible, and the first illustrations should be very simple, far simpler than this.

But can we extend this definition to all rational numbers, and say: The square root of any rational number is one of its two equal factors? Yes, provided that we first prove that every rational number is the product of two equal factors. But this proof cannot be made in the domain of rational numbers. It can be proved, for example, that the square of no rational number is two. It can also be proved that a line segment exists which is the side of a square whose area is 2 square units. This segment is the geometrical square root of 2. It can further be proved that this segment and the unit segment have no common measure. Consequently, if we try to define $\sqrt{2}$ as the numerical measure of the segment, we offer as definition that which, by proof, does not exist.

If we take the bull by the horns and *posit* the existence of a number whose product by itself is 2, we have made little advance. This mysterious number, that we have supposed to exist but have never been able to catch and examine, how shall it be multiplied by itself, or by any other number? Until we know something about the number, how can we know anything about operations on it? But even supposing that we are satisfied with knowing the *result* of the unknown operation, namely, 2, what have we to work on when the result is not given? What is 5 times $\sqrt{2}$; $\sqrt{3}$ times $\sqrt{2}$? The use of terms already defined for rational numbers makes it easy to feel that the same meaning persists if the terms are applied to new and unknown objects. But is this warranted? Because "John strikes the ball," "The clock strikes the hour," "The choirmaster strikes the pitch," "The captain strikes the flag," "A new idea strikes me," "The men strike a bargain," all have a meaning, does it follow that "The jabberwock strikes the snark" has a meaning, so long as we do not know what the "jabberwock" and the "snark" are? Even if we suppose that there *is* a "jabberwock," how do we know that it can strike? If we make the hypothesis that it can strike, which mode of striking do we mean? Since we are simply making supposi-

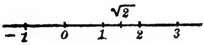
Meaning of $\sqrt{2}$ times $\sqrt{3}$ not obvious.

tions, what is to hinder us from supposing that the "jabberwock" strikes in an entirely new way?

Similarly, the fact that 2 times 3, $\frac{3}{4}$ times 12, $\frac{2}{3}$ times $\frac{5}{7}$, 3 times -5, -2 times -7, all have meanings, does not of itself give a meaning to $\sqrt{2}$ times $\sqrt{3}$.

The hypotheses at this stage may just as well be expressed thus: We suppose a number exists, call it *twosk*, and an unknown operation, call it *mang*, such that *twosk mang twosk equals two*. The mere fact that the result is *two*, and that the operator is the same as the operand, sheds little light on the nature of *twosk* or *mang*, and none whatever on the meaning of *twosk mang threesk*.

It is apparent that we have not yet succeeded in reaching a working definition of $\sqrt{2}$, and of multiplication when one factor at least is $\sqrt{2}$. This definition must be expressed in terms of what is already known, namely, the rational numbers. If n is a rational number, n times n has a meaning. Further, it has a relation to our desired result 2, namely, we can state whether or not n^2 is greater than or less than 2. (If we represent all rational numbers by points on a straight line, the point representing the geometric $\sqrt{2}$ separates the other points into two classes, the points representing numbers whose squares are less than two lying to the left, and those whose squares are greater than 2 to the right.) This leads to the following definition:



Definition of $\sqrt{2}$.

The square root of two is defined or pointed out by the partition of all rational numbers into two classes, — one class containing all rational numbers whose squares are less than two, the other all rational numbers whose squares are greater than two.

And generally:

Any partition of all rational numbers (except at most one) into two classes such that each class contains numbers, and such that every number of one class is smaller than any number of the other, defines a number.

The classes are called the smaller and the larger class respectively. This definition conforms to the standard of generalized definitions in that it includes as particular cases the numbers already defined, the rational numbers. The number 5, for example, corresponds to the partition in which one class includes all rational numbers equal to or less than 5, the other class all rational numbers greater than 5.

Definition of Equal Numbers.

Two numbers so defined are said to be *equal*, if at most one rational number is differently classified in the two definitions.

Thus 5 may be defined as above; also by a partition in which the smaller class includes all rational numbers less than 5, and the larger class all numbers equal to or greater than 5; also by a partition in which the smaller class includes all numbers less than 5, the larger class all numbers greater than 5, and 5 itself is not classified.

If the smaller class of the partition defining one of two numbers includes more than one number not contained in the smaller class of the other, the two numbers are said to be *unequal*, and the former is the *larger*.

In a partition in which all rational numbers are classified, the number defined is called *rational* if either the smaller class contains a largest number, or the larger class contains a smallest number, and this largest (or smallest) number is the number defined. Otherwise the number is called *irrational*.

Definition of Sum.

The *sum* of two numbers, a and b, is the number c defined by the classes made up as follows:

The larger class of c is made up of all possible sums of pairs of rational numbers taken one each from the larger classes of a and b; the smaller class of c is made up of all the rational numbers not in the larger class. (It would be necessary to show that if any rational number n is in the larger class, as thus defined, all rational numbers larger than n are also in the larger class, but we do not delay on this point.)

Definition of Positive and Negative Numbers.

A number is said to be *positive* when its smaller class includes all the negative rational numbers.

The number defined by the classes consisting respectively of all the negatives of the numbers in the classes defining a, is called the *negative* of a and denoted by $-a$. The *negatives* of the positive numbers are the *negative* numbers.

For subtraction, we define $a - b = a + (-b)$.

The number b is said to be less than a if $a - b$ is positive.

Definition of Subtraction and Multiplication.

If a and b are positive numbers, the definition of sum above may be used to define product by replacing the word *sum* by *product* wherever it occurs.

For the other cases of multiplication, we define:

$$(-a)\,b = -ab$$
$$a\,(-b) = -ab$$
$$(-a)\,(-b) = ab,$$

where a and b are positive numbers.

How to prove $\sqrt{2}\cdot\sqrt{3} = \sqrt{6}$.

By means of these definitions it is possible to prove that $\sqrt{2}\cdot\sqrt{3} = \sqrt{6}$. It would be necessary and sufficient to show:

(1) That the product of any rational number whose square is greater than 2, by any rational number whose square is greater than 3, is a number whose square is greater than 6;

(2) Every rational number whose square is greater than 6 is the product of two rational numbers whose squares are respectively greater than 2 and 3.

It is not necessary to take up this proof here.

What has been said will suffice to prove the following important theorem:

An Important Theorem.

Given an unending sequence of rational numbers such that:

(1) *Each number is larger than any that precedes it;*

(2) *No number of the sequence exceeds a fixed rational number in value;*

Then this sequence of numbers may be used to define a number by the method explained above.

For, for every rational number n, there either exists a number of the sequence that is larger than it or there does not. All rational numbers may be distributed into two classes according as there are or are not numbers of the sequence larger than they. There will be numbers in each class, and all numbers in the first class are smaller than any number of the second class. This partition, therefore, defines a number.

This theorem is a precise wording of the following more customary form:

If a variable quantity is constantly increasing, but does not grow large without bound, it approaches a limit.

The strict definition of limit previously given may be extended also to irrational numbers as now defined, for the definition demands simply that *difference* and the relations of "less than" and "greater than" be defined for the numbers spoken of in the definition.

Definition of Limits extended to Irrational Numbers.

According to the definition of a limit, the number defined by the sequence of rational numbers of the theorem above is the *limit* of the numbers of the sequence regarded as values assumed by some variable.

For example, having defined $\sqrt{2}$, and having defined irrational limits, we find as a *result* that $\sqrt{2}$ is the limit of the sequence:

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \ldots$$

We are now ready to attack the problem of the area of the circle. Considering the areas of any unending sequence of inscribed regular polygons, of which each one has more sides than any that precedes it, we have a sequence falling under the theorem above, and consequently pointing out a limit. There may be many such sequences, but it can be proved that they point out the same limit. This limit we call the area of the circle.

The Area of the Circle.

All that has been said is readily modified to cover the case of a sequence of decreasing values.

Another theorem commonly taken up is the following, and more or less elaborate "proofs" are given for it:

"If two variable quantities are always equal and each approaches a limit, these limits are equal."

Limit of two Equal Variables.

If the two variables are always equal, they are the same, though they may be designated by two different names.[3] The essential thing is to find out whether the one variable approaches a limit under the conditions of the problem. Thus the essential question in considerations concerning the area of the circle is whether the sequence of numbers expressing the areas of the inscribed regular polygons has a limit. The customary treatment busies itself with a matter of notation or nomenclature, and does not touch upon the real point in question.

Another source of difficulty in understanding limits may be seen in the failure to recognize that the *limit* is a constant quantity. Its value is pointed out by a series of varying quantities, but the limit itself is a fixed quantity.

The Limit a Constant.

It is lack of clearness on this point that permits setting up of theorems like:

"If the difference between two quantities *A* and *B* can be made smaller than any assigned quantity, then the quantities *A* and *B* are equal."

The difference between *A* and *B* cannot be changed. It is fixed once for all, by the nature of the quantities themselves. But we may not know what this difference is, and may find out about it by means of some other quantities which can vary.

This theorem is often applied, for example, in the demonstration of the proposition that if two pyramids have equal altitudes and bases of equal area, their volumes are equal.

The volumes of the two pyramids are either equal or not from the very beginning. If not equal, they have a fixed dif-

[3] It will be recalled that the variables considered in geometric limiting processes are numbers, the numerical measures of the magnitude of the geometric variables. Thus, when we say: "The area of a regular polygon is one half the product of its perimeter and apothem, we mean that the *number* of square units in this area is the same number as the product of the numbers of linear units in the perimeter and the apothem.

ference which never changes. It cannot be "made small" or "made large," or made any different from what it *is*. But by considering certain sets of prisms whose volumes do vary, we infer that this difference must be zero, and hence that the two pyramids must be of equal volume.

The preceding Sketch suggestive merely.

The above sketch is far from constituting a complete treatment of the subject; it is intended merely to bring out some essential points that must be considered somehow in a rigorous treatment, and to indicate very summarily one mode of doing so. It is not expected that one who meets these ideas for the first time here will get a satisfactory knowledge of the subject from the fragments given; those interested are referred to the works which deal more fully with the subject.[4] If the reader sees the need of such considerations and realizes how small is the content of the so-called proofs by limits usually given in elementary geometry, the purpose of this discussion will have been achieved.

Class-room Treatment.

What is the bearing of all this on actual teaching? At least two courses are open to the teacher. On the one hand, he may give a strict treatment. Few readers will fail to agree that it would be out of place to attempt this in our secondary schools, though several text-books intended for school use have embodied treatment along strict lines.[5]

It is evident, at least, that no teacher should attempt to teach these things to pupils unless he feels sure that he understands them thoroughly himself, and that he can present them so clearly that the pupils will understand and appreciate them. On the other hand, he may adopt the suggestion made above, and prove only commensurable cases, when such exist, and when not, establish concretely the approximate correctness of

[4] For example (in English): Dedekind, *Essays on Numbers* (transl. Beman), Chicago, 1901; Fine, *Number System of Algebra*, Boston, 1890; Fine, *College Algebra*, Boston, 1904.

[5] For example: Tannery, J., *Arithmétique*, Paris, 1900; Hadamard, *Géométrie élémentaire*, Paris, 1898; Faifofer, *Geometria*, Venice, 13th ed., 1900.

the formula. This would be in consonance with the customary treatment of irrational numbers in elementary algebra, which assumes the existence of irrational numbers, and the laws of calculation with them as symbols, and proceeds to apply these laws in practice, using approximate rational values whenever the question of value comes up.

Is there a middle ground? Of that each teacher must judge for himself. But every such middle course should have certain characteristics. While assumptions may be made at will, something should be left to prove, and the need for the proof and its cogency should be appreciated by the pupil as well as the teacher. In addition, the proof should, of course, be simple enough not to require an expenditure of time and energy upon it disproportional to the results attained.

Infinity.

Before leaving this subject a few words may be said concerning the term "infinity." Although one could get on very well in elementary mathematics without it, this term is frequently used there, and a brief discussion of it is in place here.

The sequence of numbers, 1, 2, 3, 4, 5, 6, ... has the property that after every number of the sequence there comes another, made by adding one to the preceding. The series has no last term. For if any term is supposed to be the last, a further term of the series can be produced by adding one to it. The series is, therefore, unending, or, expressed in Latin, the series is *infinite*. This, however, means nothing more than what has just been said, namely, that after every number of the series there comes another, or, negatively, that the series is *unending*.

The Series of Integers Infinite.

Whenever any set of objects has the same property, we say that the set is infinite, or sometimes that the number of objects in the set is infinite. This means simply that if we try to number or count the objects in the set, we meet the same state of affairs which exists in the series of positive integers, namely, that no matter how many have been taken there still remain others. Such, for example, is the series of inscribed regular polygons, each of which has twice as many sides as the preceding one; or the series of rational fractions of numerator unity, etc. To speak of ending such a series in any way whatsoever is a contradiction in terms, since by definition the series has no end. It would be such a contradiction, for example, to call the circle the last of the series of inscribed polygons.

In analytic geometry, in some phases of algebra, and in other branches of mathematics, the term "infinite" (or infinity) is used with a slight addition to the meaning just indicated. For example, we say that parallel straight lines intersect at infinity. What does this mean?

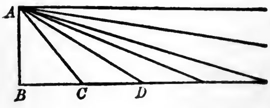
By definition, parallel straight lines are unterminated straight lines in the same plane which do not intersect. If "infinity" is the name of any place or point, the definition asserts that the straight lines do not intersect there. If we find two straight lines intersecting at any point, even though it be named infinity, they are not lines of the sort that we intend to call *parallel.*

But the term "infinity" as used above is not meant to name a point. What information does the statement then convey? It states two things:

Intersection of Parallel Straight Lines.

(1) Parallel straight lines do not intersect.

(2) If two non-parallel straight lines in the same plane be moved so as to become more and more nearly parallel (for example, by revolving one about a fixed point in it), the point of intersection moves farther and farther away, and can be moved far away at will by taking the lines sufficiently nearly parallel.



Thus, in the figure, if the line *A C* turns about the point *A*, there is no bound (infinitude) to the distance from *B* to which the point of intersection can be moved by taking the line sufficiently near to the parallel position.

Division by Zero.

Similarly, $\frac{1}{0}$ is sometimes said to be infinity, or in symbols, $\frac{1}{0} = \infty$. What does this mean? Division is the process of finding a number (the quotient) such that the product of quotient and divisor is the dividend. The form $\frac{12}{4}$ is a symbolic way of proposing the problem: To find a number x such that $4x$ is 12, and the proof that $\frac{12}{4}=3$, is that $4 \cdot 3=12$. Consequently, $\frac{1}{0}$ is a symbolic way of proposing the problem: To find a number x, such that 0 times x is 1.

But we know that zero times any number is zero. Consequently, no such number x exists. As in the case of parallel lines, the statement $\frac{1}{0}=\infty$ is used to abridge two statements: First, the symbol $\frac{1}{0}$ has no value, according to definition (of division). Second, the symbol $\frac{1}{y}$ has a definite value whenever y is not zero, and this value can be made large at will by taking y sufficiently near to zero.

Such abridged expressions are convenient because they obviate distinctions into cases. We say indiscriminately "all straight lines intersect." Their point of intersection can be specified if the lines are not parallel. In the case of parallel straight lines the technical expression "at infinity," as explained above, states what we know relative to the intersection of parallel straight lines, and its acceptance enables us to treat all straight lines without exception as intersecting. This is decidedly convenient in analytic geometry. For example, suppose that the relative position of two straight lines is sought, and that it is convenient to do this by means of their point of intersection. The two lines might be parallel, but without concern as to that, we proceed to seek the co-ordinates of their point of intersection. If they are parallel, what we know about these co-ordinates will be expressed by the same term that expresses what we know about the intersection of parallel lines, namely, the term *infinity*. The algebraic work, though leading to no numerical solution, tells us definitely what geometric condition exists — parallelism. This result comes as a matter of course in the calculation without any previous thought as to whether or not the lines are parallel.

Utility of the Term "Infinity."

In more elementary mathematics this use of the term "infinity" is hardly needed except for the trigonometric ratios of certain angles. For example, we say that tan $90° = \infty$ in the sense explained above, and this nomenclature is convenient in seeking an angle by means of its tangent. If the angle is a right angle, we find in the numerical calculation a value for the tangent which conforms to the conditions that we characterized above by the term "infinity."

When needed.

It must not be forgotten that the term "infinity" is taken from the Latin, and that if correctly used it expresses, either in a full or an abridged form, a meaning which could be expressed just as well by the use of the terms "without bound," "boundless," "boundlessly," "without end," etc.

In elementary mathematics there is little need for the term or the idea before trigonometry. Even there the need for it is slight.

To summarize: We have seen the nature of the idea of a limit, its difficulty, the inadequacy of the ordinary treatment in geometry, and the complete overlooking of essentially the same problem in algebra (irrational numbers). It has been suggested that in the class-room the practice be continued of taking for granted the existence and properties of irrational numbers, of seeing that these numbers are pointed out approximately by rational numbers, and of using these rational approximations in calculations; it was further suggested that what is essentially the same problem should be treated in the same way in geometry, by confining proofs to commensurable cases when there are such, and of assuming the relations on the basis of approximate determinations in the other cases. The ideas of successive approximation, of approximation close at will, of a sequence of values of a variable quantity pointing out some definite number, of a variable quantity whose values may grow large without bound will all be developed in this work naturally and informally without any apparatus of a "theory of limits," with the common parlance use of the term "limit," but without any definition of it as a mathematical term. Such treatment may be expected to be fully as satisfactory to the pupil as a more elaborate theory which he cannot really understand; it will

Summary.

economize his time and energy, and permit him to secure by so much the larger and better an acquaintance with the more useful parts of the subject, while the few who go on into additional mathematics will have a good foundation, free from errors and misconceptions, on which the later and more technical treatment of limits can be profitably built up.

SUPPLEMENTAL BIBLIOGRAPHY

Covering the Period 1906–1913

I. **The International Commission on the Teaching of Mathematics.** By far the most important single event on the field of the teaching of mathematics in the period of time under consideration was the appointment by the Fourth International Congress of Mathematicians, held at Rome in 1908, of an International Commission to make a comparative study of the teaching of mathematics in the different nations. The Commission was made up of forty-three members, representing twenty-four countries, and submitted its report to the Fifth International Congress of Mathematicians held at Cambridge, England, in 1912, in the form of more than 280 reports in at least 150 volumes and pamphlets, aggregating over 9000 pages. About 50 additional reports were in course of preparation at the time of the Congress.

This vast body of material gives a full and authoritative account of the present-day conditions and tendencies in the teaching of mathematics in all the leading nations. A complete list of the reports submitted is given in the "Proceedings of the Fifth International Congress of Mathematicians," Cambridge, 1913, pp. 642–653.

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No. 34. Mathematical Examinations at Oxford. By Mr. **A.-L. Dixon.** 117 pp. Price, 16*d.*

These reports have been collected in two volumes under the title: *The Teaching of Mathematics in the United Kingdom.* Part I. and Part II. Price: Vol. I., 3*s.*; Vol. II., 1*s.* 9*d.*

The report of Japan, in two volumes of 550 and 238 pages, is also published in English.

The report of Germany is by far the most thorough and extensive of all, and is to consist of 36 treatises (25 of them submitted at Cambridge), to be had separately, but planned for combination in five large volumes (Publisher, B. G. Teubner, Leipzig). The non-German will perhaps find the best general view of German conditions and methods in the report by Lietzmann, *Die Organisation des mathematischen Unterrichts an den höheren Knabenschulen in Preussen.* Pp. viii + 204. 1910. Price, 5 Marks.

A sketch of the German reports has been published under the title, *The Present Teaching of Mathematics in Germany*, by David Eugene Smith and others, TEACHERS COLLEGE RECORD, Columbia University, New York, March, 1912.

The Congress at Cambridge directed the Commission to continue its work and to submit further reports at the Sixth International Congress of Mathematicians to be held at Stockholm, Sweden, in 1916. Numerous publications in the interim will doubtless be a result of this work, and may be expected to be attainable under the same conditions as the publications of the previous quadrennium.

II. **Bibliography**. An excellent bibliography of the Teaching of Mathematics, for the period 1900–1912, by David Eugene Smith and Charles Goldziher, has been published by the Bureau of Education. The bibliography contains 1849 entries classified under eighteen headings.

An annotated list of 40 mathematical titles is contained in Bulletin No. 1, 1911, of the Bureau of Education, entitled "Bibliography of Science Teaching."

The Bureau of Education also publishes a "Monthly Record of Current Educational Publications," with annotations, indexing books, and the articles of a large number of American and European periodicals.

All of the above can be obtained free within the United States, on request addressed to the Bureau of Education, Washington, D. C.

III. **Journals**. The *Mathematics Teacher*, published quarterly at Syracuse, N. Y., has been added to the list of journals devoted to the teaching of secondary mathematics. The *American Mathematical Monthly* has changed its management, policy, and form, and is now devoted chiefly to collegiate mathematics. *L'Enseignement Mathématique*, in view of its character as an international journal, has recently abandoned its exclusive use of the French language, and will hereafter publish papers and other communications in any one of the four official languages of the International Congress of Mathematicians, — English, French, German, Italian.

All the existent journals devoted wholly or in considerable part to the teaching of mathematics have continued their activities, and have published many excellent papers. Space is too limited to permit a listing of their titles here. The reader is referred to the bibliography by Smith and Goldziher, and to the files of the journals themselves.

IV. **Books and other Independent Publications**. Referring to the bibliography of Smith and Goldziher, for the titles of works in languages other than English, we mention here only some of the more important works in English that have appeared in the period in question.

Proceedings of the Fifth International Congress of Mathematicians. Cambridge, 1913. Vol. II. (pp. 447–653) contains the Communications presented to the sections on Philosophy, History and Didactics of Mathematics.

Monographs on Topics of Modern Mathematics Relevant to the Elementary Field, edited by J. W. A. Young. New York, Longmans, Green & Co., 1911, pp. viii + 316. The monographs included in the volume are as follows: —

1. The Foundations of Geometry. By OSWALD VEBLEN, Princeton University.

2. Modern Pure Geometry. By THOMAS F. HOLGATE, Northwestern University.

3. Non-Euclidean Geometry. By FREDERICK S. WOODS, The Massachusetts Institute of Technology.

4. The Fundamental Propositions of Algebra. By EDWARD V. HUNTINGTON, Harvard University.

5. The Algebraic Equation. By G. A. MILLER, The University of Illinois.

6. The Function Concept and the Fundamental Notions of the Calculus. By GILBERT AMES BLISS, The University of Chicago.

7. The Theory of Numbers. By J. W. A. YOUNG, The University of Chicago.

8. Constructions with Ruler and Compasses; Regular Polygons. By L. E. DICKSON, The University of Chicago.

9. The History and Transcendence of π. By DAVID EUGENE SMITH, Teachers College, Columbia University.

This work seeks to bring within reach of secondary teachers (in service or in training), college students, and others at a like stage of mathematical advancement, a scientific treatment of some of the regions of advanced mathematics that have points of contact with the elementary field. The amount of mathematical knowledge presupposed on the part of the reader varies with the different subjects. A large part of the book presupposes only knowledge of elementary geometry and algebra, together with a certain measure of mathematical maturity. On the other hand, there is much that will repay careful and detailed study by advanced students. So far as the subject-matter permits, the less difficult topics are taken up first in each monograph.

Whitehead, A. N. An Introduction to Mathematics. London and New York, 1911, pp. 255.

An exposition of numerous fundamental concepts of mathematics in simple, "popular" style.

Young, J. W. Fundamental Concepts of Algebra and Geometry. New York, 1911, pp. vii + 247.

"An elementary account of the logical foundations of algebra and geometry, giving a general exposition of the abstract, formal point of view developed during the last few decades."

Evans, G. W. The Teaching of High School Mathematics. Boston, 1911, pp. ix + 93.

Schultze, A. The Teaching of Mathematics in Secondary Schools. New York, 1912, pp. xx + 370.

Branford, B. A Study of Mathematical Education including the Teaching of Arithmetic. Oxford, 1908, pp. xii + 392.

Lodge, Sir Oliver. Easy Mathematics. London and New York, 1905. Pp. 436.

Stamper, A. W. The History of the Teaching of Elementary Geometry with Reference to Present-Day Problems. New York, 1909.

Smith, David Eugene. The Teaching of Geometry. Boston, 1911.

Jackson, L. L. The Educational Si[illegible]nce of Sixteenth Century Arithmetic. New York, 1906.

McMurry, C. A. Special Meth[illegible], pp. vii + 225.

Stone, C. W. Arithmetical Abilities and Some Factors in Determining Them. New York, 1908.

Smith, David Eugene. The Teaching of Arithmetic. New York, 1911, pp. 100.

Suzallo, H. The Teaching of Primary Arithmetic. A critical study of recent tendencies in method. With an introduction by **D. E. Smith.** Boston, 1911, pp. x + 123.

Stamper, A. W. The Teaching of Arithmetic. New York, 1913, pp. 284.

Smith, David Eugene. *Rara Arithmetica.* Boston, 1908, pp. 507.

A catalogue of the arithmetics written before 1601. Contains numerous descriptions and historical notes as well as nine plates, and 246 figures, usually full-page facsimile reproductions. A work of the first importance for those studying this branch of the history of mathematics.

Smith, David Eugene, and others. Number Games and Number Rhymes. TEACHERS COLLEGE RECORD. November, 1912, pp. 111.

White, Wm. F. Scrapbook of Elementary Mathematics. Chicago, 1908.

Contains recreations, curiosities, quotations, etc.

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